Continuous Optimisation: Chpt 1 Exercises

November 18, 2015

1. Show that $\mathcal{A}$ is a convex cone if and only if $\lambda_1 x + \lambda_2 y \in \mathcal{A}$ for all $\lambda_1, \lambda_2 > 0$ and all $x, y \in \mathcal{A}$.

**Solution:**

$(\Leftarrow)$ For all $x, y \in \mathcal{A}$ and $0 < \theta < 1$, letting $\lambda_1 = \theta > 0$, $\lambda_2 = 1 - \theta > 0$ we have $\theta x + (1 - \theta)y = \lambda_1 x + \lambda_2 y \in \mathcal{A}$ and thus $\mathcal{A}$ is convex.

For all $z \in \mathcal{A}$ and $\mu > 0$, letting $x = y = z$ and $\lambda_1 = \lambda_2 = \frac{1}{\mu}$ we have $\mu z = \lambda_1 x + \lambda_2 y \in \mathcal{A}$ and thus $\mathcal{A}$ is a cone.

$(\Rightarrow)$ Consider arbitrary $\lambda_1, \lambda_2 > 0$ and $x, y \in \mathcal{A}$. Let $\mu = \lambda_1 + \lambda_2$ and $\theta = \lambda_1 / \mu$. We have $0 < \theta < 1$ and as $\mathcal{A}$ is convex $\theta x + (1 - \theta)y \in \mathcal{A}$. As $\mathcal{A}$ is a cone, we have $\mu (\theta x + (1 - \theta)y) \in \mathcal{A}$, and noting that $\mu (\theta x + (1 - \theta)y) = \lambda_1 x + \lambda_2 y$, this completes the proof.

2. For arbitrary $a \in \mathbb{R}^n$ show that the set $\mathcal{A} = \{x \in \mathbb{R}^n \mid \langle a, x \rangle \geq 0\}$ is a convex cone.

**Solution:** Consider arbitrary $x, y \in \mathcal{A}$ and $\lambda_1, \lambda_2 > 0$. We have $\langle a, x \rangle \geq 0$ and $\lambda_1 > 0$, which implies that $\lambda_1 \langle a, x \rangle \geq 0$. Similarly $\langle a, y \rangle \geq 0$ and $\lambda_2 > 0$ implies that $\lambda_2 \langle a, y \rangle \geq 0$. Therefore $\langle a, (\lambda_1 x + \lambda_2 y) \rangle = \lambda_1 \langle a, x \rangle + \lambda_2 \langle a, y \rangle \geq 0$, which implies that $\lambda_1 x + \lambda_2 y \in \mathcal{A}$, and thus $\mathcal{A}$ is a convex cone.

3. Show that the intersection of two convex cones is itself a convex cone.

**Solution:** Let $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathbb{R}^n$ be convex cones and consider $\mathcal{K} = \mathcal{K}_1 \cap \mathcal{K}_2$.

Consider arbitrary $x, y \in \mathcal{K}$ and $\lambda_1, \lambda_2 > 0$. For $j = 1, 2$ we have $x, y \in \mathcal{K}_j$, $\lambda_1, \lambda_2 > 0$ and thus $\lambda_1 x + \lambda_2 y \in \mathcal{K}_j$. This implies that $\lambda_1 x + \lambda_2 y \in \mathcal{K}_1 \cap \mathcal{K}_2 = \mathcal{K}$, and thus $\mathcal{K}$ is a convex cone.
4. Show that for $\mathcal{A} \subseteq \mathbb{R}^n$ the set $\text{conic}(\mathcal{A})$ is the smallest convex cone containing $\{0\} \cup \mathcal{A}$.

**Solution:** If $\mathcal{A} = \emptyset$ then $\text{conic}(\mathcal{A}) = \{0\}$, which is trivially the smallest convex cone containing $\{0\} \cup \mathcal{A}$. From now on we consider when $\mathcal{A} \neq \emptyset$.

From the definition we trivially have $\{0\} \cup \mathcal{A} \subseteq \text{conic}(\mathcal{A})$.

We will now show that $\text{conic}(\mathcal{A})$ is a convex cone. Consider an arbitrary $x, y \in \text{conic}(\mathcal{A})$ and $\lambda_1, \lambda_2 > 0$. We have $x = \sum_{i=1}^{k} \lambda_i x^i$ and $y = \sum_{i=1}^{l} \nu_i y^i$ for some $k, l \geq 0$, $\omega^i, \nu^i \geq 0$, $x^i, y^i \in \mathcal{A}$. Therefore

$$\lambda_1 x + \lambda_2 y = \sum_{i=1}^{k} (\lambda_1 \omega^i) x^i + \sum_{i=1}^{l} (\lambda_2 \nu^i) y^i \in \text{conic}(\mathcal{A}).$$

We will now consider an arbitrary convex cone $K \subseteq \mathbb{R}^n$ such that $\{0\} \cup \mathcal{A} \subseteq K$ and we shall show that $\text{conic}(\mathcal{A}) \subseteq K$. First we will give an alternative definition of the conic hull, noting that as $\mathcal{A} \neq \emptyset$ we can then restrict $k$ in the definition to be strictly positive. We have:

$$\text{conic}(\mathcal{A}) = \left\{ \begin{array}{c} x = \sum_{i=1}^{k} \lambda^i x^i \mid x^i \in \mathcal{A}, \quad \lambda^i \geq 0, \quad k > 0 \end{array} \right\}$$

$$= \left\{ \begin{array}{c} x = \mu \sum_{i=1}^{k} \theta^i x^i \mid x^i \in \mathcal{A}, \quad \mu \geq 0, \quad \theta^i \geq 0, \quad \sum_{i=1}^{k} \theta^i = 1, \quad k > 0 \end{array} \right\}$$

(Letting $\mu = \sum_{i=1}^{k} \lambda^i$ and $\theta^i = \lambda^i / \mu$)

$$= \left\{ \begin{array}{c} x = \mu y \mid \mu \geq 0, \quad y \in \text{conv}(\mathcal{A}) \end{array} \right\}$$

As $\mathcal{A} \subseteq K$ and $K$ is convex, we have $\text{conv}(\mathcal{A}) \subseteq K$. As $\text{conv}(\mathcal{A}) \subseteq K$ and $K$ is a cone which contains $0$, for all $\mu \geq 0$ and $y \in \text{conv}(\mathcal{A})$ we have $\mu y \in K$. Therefore $\text{conic}(\mathcal{A}) \subseteq K$, which completes the proof.

5. Show that $\mathbb{R}_+^n$ is pointed.

**Solution:** Suppose for the sake of contradiction we have $\pm x \in \mathbb{R}_+^n \setminus \{0\}$.

As $x \in \mathbb{R}_+^n$ we have $x_i \geq 0$ for all $i$.

As $-x \in \mathbb{R}_+^n$ we have $-x_i \geq 0$ for all $i$.

Therefore $x_i = 0$ for all $i$.

This gives the contradiction $(x_0, x) = (0, 0)$. 

Page 2
6. For arbitrary norm $\| \cdot \|$, show that the following set is pointed:
$$ A = \{(x_0, x) \in \mathbb{R} \times \mathbb{R}^n \mid \|x\| \leq x_0\}. $$

**Solution:** Suppose for the sake of contradiction we have $\pm(x_0, x) \in A \setminus \{(0, 0)\}$. As $(x_0, x) \in A$ we have $x_0 \geq \|x\| \geq 0$. As $-(x_0, x) \in A$ we have $-x_0 \geq \|-x\| \geq 0$. Therefore $x_0 = 0$, which in turn implies that $\|x\| = 0$. This gives the contradiction $(x_0, x) = (0, 0)$.

7. Show that $A = \{(x_0, x) \in \mathbb{R} \times \mathbb{R}^n \mid \|x\| \leq x_0\}$ is full-dimensional.

**Solution:** It is sufficient to show that the following $n + 1$ vectors are linearly independent:
$$(1, 0), (\|e_1\|, e_1), \ldots, (\|e_n\|, e_n) \in A.$$ 
Suppose for the sake of contradiction this is not the case. Then there exists $(\theta_0, \theta) \in \mathbb{R} \times \mathbb{R}^n \setminus \{(0, 0)\}$ such that
$$(0, 0) = \theta_0(1, 0) + \sum_{i=1}^n \theta_i(\|e_i\|, e_i) = (\theta_0 + \sum_{i=1}^n \theta_i \|e_i\|, \theta).$$
We thus get the contradiction $\theta = 0$ and $\theta_0 = -\sum_{i=1}^n \theta_i \|e_i\| = 0$.

8. Show that if $K_1 \subseteq \mathbb{R}^n$ and $K_2 \subseteq \mathbb{R}^m$ are proper cones then $K_1 \times K_2 := \{(x, y) \mid x \in K_1, y \in K_2\}$ is a proper cone.

**Solution:** We will split this into showing that $K_1 \times K_2$ satisfies all the separate conditions of being a proper cone:

1. **Closed:** Consider $(x, y) \in \text{cl}(K_1 \times K_2)$. We have $(x, y) = \lim_{i \to \infty} (x^i, y^i)$ for some $\{ (x^i, y^i) \mid i \in \mathbb{N} \} \subseteq K_1 \times K_2$. We have $x = \lim_{i \to \infty} x^i$ and $x^i \in K_1$ for all $i$, and as $K_1$ is closed this implies that $x \in K_1$. Similarly $y \in K_2$, and thus $(x, y) \in K_1 \times K_2$, which implies that this set is closed.

2. **Convex cone:** Consider arbitrary $(x_1, y_1), (x_2, y_2) \in K_1 \times K_2$ and $\lambda_1, \lambda_2 > 0$. We want to show that $\lambda_1(x_1, y_1) + \lambda_2(x_2, y_2) \in K_1 \times K_2$, or equivalently show
that \((\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 y_1 + \lambda_2 y_2) \in K_1 \times K_2\). As \(K_1\) is a convex cone we have \(\lambda_1 x_1 + \lambda_2 x_2 \in K_1\). Similarly, as \(K_2\) is a convex cone we have \(\lambda_1 y_1 + \lambda_2 y_2 \in K_2\). Therefore \((\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 y_1 + \lambda_2 y_2) \in K_1 \times K_2\).

3. Pointed: Suppose that \(\pm (x, y) \in K_1 \times K_2 \setminus \{(0, 0)\}\). We have \(\pm x \in K_1\) and as \(K_1\) is pointed this implies that \(x = 0\). Similarly, we also have \(y = 0\), and thus get the contradiction \((x, y) = (0, 0)\).

4. Full-dimensional: As \(K_1\) is full dimensional, there exist linearly independent vectors \(x^1, \ldots, x^n \in K_1\). Similarly, there are linearly independent vectors \(y^1, \ldots, y^m \in K_2\). We also have \(0 \in K_1\) and \(0 \in K_2\).

We now consider the \(n + m\) vectors \((x^1, 0), \ldots, (x^n, 0), (0, y^1), \ldots, (0, y^m) \in K_1 \times K_2\). If we can show that these are linearly independent then we are done. Suppose for the sake of contradiction that they are not linearly independent. Then there exists \(\theta \in \mathbb{R}^{n+m} \setminus \{0\}\) such that

\[
(0, 0) = \sum_{i=1}^{n} \theta_i (x^i, 0) + \sum_{i=1}^{m} \theta_{n+i} (0, y^i)
= \left(\sum_{i=1}^{n} \theta_i x^i, \sum_{j=1}^{m} \theta_{n+j} y^j\right).
\]

We have \(0 = \sum_{i=1}^{n} \theta_i x^i\) and as \(x^1, \ldots, x^n\) are linearly independent, this implies \(\theta_i = 0\) for all \(i = 1, \ldots, n\). Similarly \(0 = \sum_{i=1}^{m} \theta_{n+i} y^i\) and \(y^1, \ldots, y^m\) being linearly independent implies \(\theta_{n+j} = 0\) for all \(j = 1, \ldots, m\). We thus have the contradiction \(\theta = 0\).

9. Which categories do the closed sets \(A_1, \ldots, A_5\) fall into out of:

\(\text{(a) Not a convex cone,}\)
\(\text{(b) Convex cone; Not pointed nor full-dimensional,}\)
\(\text{(c) Convex cone; Pointed but not full-dimensional,}\)
\(\text{(d) Convex cone; Full-dimensional but not pointed,}\)
\(\text{(e) Proper cone (closed convex full-dimensional pointed cone).}\)

\[A_1 = \{x \in \mathbb{R}^n \mid a_1^T x \geq 0\},\]
\[A_2 = \{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\},\]
\[A_3 = \text{conic}\{a_1, \ldots, a_n\},\]
\[A_4 = \{\lambda a_1 \mid \lambda \in \mathbb{R}\},\]
\[A_5 = \{\lambda a_1 \mid \lambda \geq 0\},\]
where \( n \geq 2 \) and \( a_1, \ldots, a_n \in \mathbb{R}^n \) linearly independent.

**Solution:**

\( A_1 \): (d) This is a convex cone which is full-dimensional but not pointed:
- From question 2 this is a convex cone.
- As \( n \geq 0 \), there exists \( b \in \mathbb{R}^n \setminus \{0\} \) such that \( b^T a_1 = 0 \), and we thus have \( \pm b \in A_1 \).
- Let \( c_1, \ldots, c_n \in \mathbb{R}^n \) such that \( c_i = \begin{cases} a_i & \text{if } a_i^T a_i \geq 0 \\ -a_i & \text{if } a_i^T a_i < 0. \end{cases} \)
  
  By construction we then have linearly independent vectors \( c_1, \ldots, c_n \in A_1 \), and thus \( A_1 \) is full-dimensional.

\( A_2 \): (a) This is not a convex cone:
For \( x = \frac{1}{n} e \) (where \( e \) is the all-ones vector), we have \( \|x\|^2 = \frac{1}{n^2} e^T e = \frac{n}{n^2} = \frac{1}{n} \leq 1 \), and thus \( x \in A_2 \). However for \( \lambda = n > 0 \) we have \( \|\lambda x\|^2 = e^T e = n > 1 \), and thus \( \lambda x \notin A_2 \).

\( A_3 \): (e) This is a proper cone:
- From question 4 this is a convex cone.
- We have the linearly independent vectors \( a_1, \ldots, a_n \in A_3 \), and thus \( A_3 \) is full-dimensional.
  - Suppose for the sake of contradiction there exists \( x \in \mathbb{R}^n \setminus \{0\} \) such that \( \pm x \in A_3 \). Then there exists \( \alpha, \beta \in \mathbb{R}^n_+ \setminus \{0\} \) such that
    
    \[
    x = \sum_{i=1}^n \alpha_i a_i, \quad -x = \sum_{i=1}^n \beta_i a_i.
    \]

  Adding these two equalities together and letting \( \gamma = \alpha + \beta \in \mathbb{R}^n_+ \setminus \{0\} \), we get \( 0 = \sum_{i=1}^n \gamma_i a_i \), which contradicts \( a_1, \ldots, a_n \) being linearly independent.

\( A_4 \): (b) This is a convex cone which is neither pointed nor full-dimensional:
- We have \( A_4 = \text{conic}\{a_1, -a_1\} \), and thus by question 4 this is a convex cone.
- As \( n \geq 0 \), there exists \( b \in \mathbb{R}^n \setminus \{0\} \) such that \( b^T a_1 = 0 \), and thus \( b^T x = 0 \) for all \( x \in A_4 \). This implies that \( A_4 \) is not full-dimensional.
- We have \( \pm a_1 \in A_4 \setminus \{0\} \) and thus \( A_4 \) is not pointed.

\( A_5 \): (c) This is a convex cone which is pointed but not full-dimensional:
- We have \( A_4 = \text{conic}\{a_1\} \), and thus by question 4 this is a convex cone.
- As \( n \geq 0 \), there exists \( b \in \mathbb{R}^n \setminus \{0\} \) such that \( b^T a_1 = 0 \), and thus \( b^T x = 0 \) for all \( x \in A_4 \). This implies that \( A_4 \) is not full-dimensional.
- Suppose for the sake of contradiction there exists \( x \in \mathbb{R}^n \setminus \{0\} \) such that \( \pm x \in A_5 \). Then there exists \( \lambda_1, \lambda_2 > 0 \) such that \( x = \lambda_1 a_1 \) and \( -x = \lambda_2 a_1 \). Therefore \( 0 = (\lambda_1 + \lambda_2) a_1 \). As \( \lambda_1 + \lambda_2 > 0 \), this implies that \( a_1 = 0 \), which contradicts \( a_1, \ldots, a_n \) being linearly independent.
10. A farmer has 20 hectares of land to be planted with either potatoes or barley or a combination of the two. The farmer is limited to 7000kg of fertiliser. Potatoes require 200kg of fertiliser per hectare, Barley requires 400kg of fertiliser per hectare. The farmer is also limited to 35kg of insecticide. Potatoes require 2kg of insecticide per hectare, Barley requires 1kg of insecticide per hectare. The farmer will get paid 900 per hectare for potatoes and 600 per hectare for barley. The farmer wants to decide how many hectare of potatoes and barley to plant in order to make as much money as possible. Formulate this problem as a linear optimisation problem.

Solution: Let $x_1$ be the amount of hectares of potatoes to plant and let $x_2$ be the amount of hectares of barley to plant. Naturally have $x \in \mathbb{R}^2_+$. Have the limit on the land of $x_1 + x_2 \leq 20$. Have the limit on the fertiliser of $200x_1 + 400x_2 \leq 7000$, or equivalently $x_1 + 2x_2 \leq 35$. Have the limit on the insecticide of $2x_1 + x_2 \leq 35$. The farmer will get paid $900x_1 + 600x_2$ euros, and we want to maximise this. Therefore the problem is

$$\max\limits_x 900x_1 + 600x_2$$

s.t. $x_1 + x_2 \leq 20$

$$x_1 + 2x_2 \leq 35$$

$$2x_1 + x_2 \leq 35$$

$x \in \mathbb{R}^2_+$.

This is equivalent to

$$\max\limits_{x,y} \left\langle \begin{pmatrix} 900 \\ 600 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} \right\rangle$$

s.t. $\left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \right\rangle = 20$, $\left\langle \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \right\rangle = 35$, $\left\langle \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \right\rangle = 35$,

$(x, y) \in \mathbb{R}^2_+ \times \mathbb{R}^3_+$. 

Page 6
11. (Adapted from Q4.14, Convex Optimization, Boyd and Vandenberghe, 2004)
Consider a system of canals which meet at \( n \) nodes. The nodes themselves cannot store water, so the amount of water flowing into them must equal the amount of water flowing out. We will let the variable \( x_{ij} \) be the amount of water we allow to flow from node \( i \) to node \( j \). The cost of this flow is \( c_{ij}x_{ij} \) (we may have to pay to pump the water uphill). Each canal has a lower bound \( l_{ij} \) and an upper bound \( u_{ij} \) for the amount of water that can flow from node \( i \) to node \( j \). An external supply of water flowing into node \( i \) is \( b_i \) (this may be negative if the water is flowing out of this node).

We wish to minimize the total cost. Formulate this problem as a linear optimization problem.

Solution:
Variables: \( x_{ij} \) such that \( i \neq j \) and \( i, j = 1, \ldots, n \).

Minimisation or Maximisation? Minimisation, as we want to minimise the cost.

Objective function: \( \sum_{i \neq j} c_{ij}x_{ij} \).

Constraint 1: \( b_i + \sum_{j \neq i} x_{ji} = \sum_{j \neq i} x_{ij} \) for all \( ij \), as the amount of water flowing into node \( i \) is equal to the amount of water flowing out.

Constraint 2: \( l_{ij} \leq x_{ij} \leq u_{ij} \) for all \( i \neq j \).

Problem in basic form:

\[
\min_x \sum_{i \neq j} c_{ij}x_{ij} \\
\text{s.t.} \quad b_i + \sum_{j \neq i} x_{ji} = \sum_{j \neq i} x_{ij} \quad \text{for all} \ i \\
\quad l_{ij} \leq x_{ij} \leq u_{ij} \quad \text{for all} \ i \neq j
\]

Slack variables:

\[
\min_{x,y,z} \sum_{i \neq j} c_{ij}x_{ij} \\
\text{s.t.} \quad b_i + \sum_{j \neq i} x_{ji} = \sum_{j \neq i} x_{ij} \quad \text{for all} \ i \\
\quad x_{ij} - y_{ij} = l_{ij} \quad \text{for all} \ i \neq j \\
\quad x_{ij} + z_{ij} = u_{ij} \quad \text{for all} \ i \neq j \\
\quad y_{ij}, z_{ij} \geq 0 \quad \text{for all} \ i \neq j
\]
Standard form: \((x_{ij} = y_{ij} + l_{ij})\)

\[
\sum_{i \neq j} c_{ij} l_{ij} + \min_{y, Z} \sum_{i \neq j} c_{ij} y_{ij}
\]

s.t. \[
\sum_{j \neq i} y_{ij} - \sum_{j \neq i} y_{ji} = b_i + \sum_{j \neq i} l_{ji} - \sum_{j \neq i} l_{ij} \quad \text{for all } i
\]
\[
y_{ij} + z_{ij} = u_{ij} - l_{ij} \quad \text{for all } i \neq j
\]
\[
y_{ij}, z_{ij} \geq 0 \quad \text{for all } i \neq j
\]

12. A new mobile mast needs to be built in order to service \(m\) villages. The coordinates of these villages are given by \(v_1, \ldots, v_m \in \mathbb{R}^2\). We consider two alternative problems for this:

1. We want to minimise the distance to the furthest village.

2. The mast cannot be more than \(R\) kilometers from any of the villages and we want to minimise the sum of the distances to all of the villages.

Formulate these two problems as second order cone optimisation problems.

Solution:

1. Basic form:

\[
\begin{align*}
\min_{t, y} & \quad t \\
\text{s.t.} & \quad \|v_i - y\|_2 \leq t \quad \text{for all } i = 1, \ldots, m
\end{align*}
\]

Second order cone: \(\mathcal{L} := \{(x_0, x) \in \mathbb{R} \times \mathbb{R}^2 \mid \|x\|_2 \leq x_0\}\)

\[
\begin{align*}
\min_{s, t, x, y} & \quad t \\
\text{s.t.} & \quad s_i \leq t \quad \text{for all } i = 1, \ldots, m \\
& \quad x^i = v_i - y \quad \text{for all } i = 1, \ldots, m \\
& \quad (s_i, x^i) \in \mathcal{L} \quad \text{for all } i = 1, \ldots, m
\end{align*}
\]
Slack variables:

\[
\min_{s,t,u,x,y} t
\]
\[
\text{s.t. } s_i + u_i = t \quad \text{for all } i = 1, \ldots, m
\]
\[
x^i + y = v_i \quad \text{for all } i = 1, \ldots, m
\]
\[
(s_i, x^i) \in \mathcal{L} \quad \text{for all } i = 1, \ldots, m
\]
\[
u \in \mathbb{R}_+^m
\]

Standard form: \((t = s_1 + u_1 \quad \text{and} \quad y = x^1 + v_1)\)

\[
\min_{s,u,x} s_1 + u_1
\]
\[
\text{s.t. } s_i + u_i - s_1 - u_1 = 0 \quad \text{for all } i = 2, \ldots, m
\]
\[
x^i - x^1 = v_i - v_1 \quad \text{for all } i = 2, \ldots, m
\]
\[
(s_i, x^i) \in \mathcal{L} \quad \text{for all } i = 1, \ldots, m
\]
\[
u \in \mathbb{R}_+^m
\]

or alternatively, from the basic form:

\[
\min_{t,y} t
\]
\[
\begin{bmatrix}
0 \\
v_1 \\
0 \\
v_m
\end{bmatrix} - t
\begin{bmatrix}
-1 \\
0 \\
-1 \\
0
\end{bmatrix}
- \sum_{i=1}^n y_i
\begin{bmatrix}
e_i \\
e_i \\
e_i \\
e_i
\end{bmatrix}
\in \mathcal{L} \times \mathcal{L} \times \cdots \times \mathcal{L},
\]

\[
\mathcal{L} \times \mathcal{L} \times \cdots \times \mathcal{L} = \left\{
\begin{bmatrix}
x_{0,1} \\
x_1 \\
\vdots \\
x_{0,m}
\end{bmatrix} \begin{bmatrix}
x_{i,1} \\
x_i \\
\vdots \\
x_{i,m}
\end{bmatrix} \begin{bmatrix}
x_{0,i} \\
x_{1,i} \\
\vdots \\
x_{m,i}
\end{bmatrix} \mid (x_{0,i}, x_i) \in \mathcal{L} \text{ for all } i = 1, \ldots, m
\right\}
\]
2. Basic form:

$$\min_{t, y} \sum_{i=1}^{m} t_i$$

s.t. $\|v_i - y\|_2 \leq t_i$ for all $i = 1, \ldots, m$

$t_i \leq R$ for all $i = 1, \ldots, m$

Standard form:

$$\min_{t, y} \sum_{i=1}^{m} t_i$$

s.t. 

$$\begin{pmatrix} R1_m \\ 0 \\ v_1 \\ \vdots \\ v_m \end{pmatrix} - \sum_{i=1}^{m} t_i \begin{pmatrix} e_i \\ -e_i \\ \vdots \\ 0 \end{pmatrix} - \sum_{j=1}^{n} y_j \begin{pmatrix} 0 \\ 0 \\ e_j \\ \vdots \end{pmatrix} \in K,$$

where $K = \left\{ \begin{pmatrix} u \\ w \\ z_1 \\ \vdots \\ z_m \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{mn} \bigg| u \in \mathbb{R}_+^m, \quad \|z_i\|_2 \leq w_i \text{ for all } i = 1, \ldots, m \right\}$,

$1_m = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^m.$