1. Show that if $A_1 \subseteq A_2$ then $A_1^* \supseteq A_2^*$.

**Solution:** Consider an arbitrary $y \in A_2^*$. From the definition of the dual we have $\langle x, y \rangle \geq 0$ for all $x \in A_2$.

For all $z \in A_1$, we have $z \in A_2$ and thus $\langle z, y \rangle \geq 0$. Therefore $y \in A_1^*$.

2. Prove that for $A \subseteq \mathbb{R}^n$ we have $(\text{conic } A)^* = A^*$.

**Solution:** We have $A \subseteq \text{conic } A$ and thus $A^* \supseteq (\text{conic } A)^*$. We are now left to show that $A^* \subseteq (\text{conic } A)^*$.

Suppose for the sake of contradiction there exists $y \in A^* \setminus (\text{conic } A)^*$. Then $\exists x \in \text{conic } A$ such that $\langle y, x \rangle < 0$.

As $x \in \text{conic } A$, we have $x = \sum_{i=1}^{k} \lambda_i a_i$ for some $\lambda_1, \ldots, \lambda_k \geq 0$ and $a_1, \ldots, a_k \in A$.

As $y \in A^*$ we have $\langle y, a_i \rangle \geq 0$ for all $i$. This then gives the contradiction $0 > \langle y, x \rangle = \sum_{i=1}^{k} \lambda_i \langle y, a_i \rangle \geq 0$.

3. Show that for all $A \subseteq \mathbb{R}^n$ we have $A \subseteq A^{**}$.

**Solution:** Suppose for the sake of contradiction there exists $x \in A \setminus (A^{**})$. As $x \notin A^{**}$, there must exist a $y \in A^*$ such that $\langle x, y \rangle < 0$. However as $x \in A$ and $y \in A^*$ we have the contradiction that $\langle x, y \rangle \geq 0$. 
4. Prove that if \( K_1, K_2 \subseteq \mathbb{R}^n \) are cones which contain the origin then \((K_1 + K_2)^* = K_1^* \cap K_2^*\).

**Solution:** We will first prove that \((K_1 + K_2)^* \subseteq K_1^* \cap K_2^*\) by considering an arbitrary \( x \in (K_1 + K_2)^* \) and showing that we then have \( x \in K_1^* \cap K_2^* \). For all \( y \in K_1 \) we have \( y = (y + 0) \in K_1 + K_2 \) and thus \( \langle y, x \rangle \geq 0 \), which implies that \( x \in K_1^* \). Similarly, for all \( z \in K_2 \) we have \( z = (0 + z) \in K_1 + K_2 \) and thus \( \langle z, x \rangle \geq 0 \), which implies that \( x \in K_2^* \). Therefore we have \( x \in K_1^* \cap K_2^* \), as required.

We will now prove that \( K_1^* \cap K_2^* \subseteq (K_1 + K_2)^* \) by considering an arbitrary \( x \in K_1^* \cap K_2^* \) and showing that we then have \( x \in (K_1 + K_2)^* \). Considering an arbitrary \( u \in K_1 + K_2 \), there exists \( y \in K_1 \) and \( z \in K_2 \) such that \( u = y + z \). As \( x \in K_1^* \), we have \( \langle y, x \rangle \geq 0 \). Similarly, as \( x \in K_2^* \), we have \( \langle z, x \rangle \geq 0 \). Therefore \( \langle u, x \rangle = \langle y, x \rangle + \langle z, x \rangle \geq 0 \). As \( u \in K_1 + K_2 \) was arbitrary, this implies that \( \langle u, x \rangle \geq 0 \) for all \( u \in K_1 + K_2 \), and thus \( x \in (K_1 + K_2)^* \).

5. Prove that a closed convex cone \( K \) is full-dimensional if and only if \( K^* \) is pointed.

**Hint:** Show the equivalent result that \( K \) is not full-dimensional if and only if \( K^* \) is not pointed, and use the 3rd condition for being full-dimensional.

**Solution:** We shall show the equivalent result that \( K \) is not full-dimensional if and only if \( K^* \) is not pointed. We have

\[
\text{\( K \) is not full dimensional} \quad \Leftrightarrow \quad \exists y \in \mathbb{R}^n \setminus \{0\} \text{ s.t.} \quad \langle x, y \rangle = 0 \quad \forall x \in K
\]

\[
\Leftrightarrow \quad \exists y \in \mathbb{R}^n \setminus \{0\} \text{ s.t.} \quad \pm y \in K^*
\]

\[
\Leftrightarrow \quad K^* \text{ is not pointed.}
\]
6. Show that the simplified dual problem to the primal problem that you formulated in Ex 1.10 is:

\[
\begin{align*}
\min_{\mathbf{u}} & \quad 20u_1 + 35u_2 + 35u_3 \\
\text{s.t.} & \quad 900 \leq u_1 + u_2 + 2u_3 \\
& \quad 600 \leq u_1 + 2u_2 + u_3 \\
& \quad \mathbf{u} \in \mathbb{R}_+^3.
\end{align*}
\]

Consider the points \( \mathbf{x}^* = (15, 5) \) and \( \mathbf{u}^* = (300, 0, 300) \). Show that these are optimal solutions to the primal and dual problems respectively.

**Solution:** In Ex. 1.8 you formulated the problem

\[
\begin{align*}
\max_{\mathbf{x}} & \quad 900x_1 + 600x_2 \\
\text{s.t.} & \quad x_1 + x_2 \leq 20 \\
& \quad x_1 + 2x_2 \leq 35 \\
& \quad 2x_1 + x_2 \leq 35 \\
& \quad \mathbf{x} \in \mathbb{R}_+^2.
\end{align*}
\]

In the standard form this is

\[
\begin{align*}
- \min_{\mathbf{x}, \mathbf{y}} & \quad \begin{pmatrix} -900 \\ -600 \\ 0 \\ 0 \\ 0 \end{pmatrix} \left< \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} \right> \\
\text{s.t.} & \quad \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} \left< \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} \right> = 20 \\
& \quad \begin{pmatrix} 1 & 2 & 0 & 0 \end{pmatrix} \left< \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} \right> = 35 \\
& \quad \begin{pmatrix} 2 & 1 & 0 & 0 \end{pmatrix} \left< \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} \right> = 35 \\
& \quad (\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^2 \times \mathbb{R}_+^3.
\end{align*}
\]
The dual to this is

$$\max_v \quad 20v_1 + 35v_2 + 35v_3$$

subject to

$$\begin{bmatrix}
-900 \\
-600 \\
0 \\
0 \\
0
\end{bmatrix} - v_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - v_2 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - v_3 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}_+^5.$$

Simplifying and substituting $u = -v$, this becomes:

$$\min_u \quad 20u_1 + 35u_2 + 35u_3$$

subject to

$$900 \leq u_1 + u_2 + 2u_3$$
$$600 \leq u_1 + 2u_2 + u_3$$
$$u \in \mathbb{R}_+^3.$$

We now consider the points $x^* = (15, 5)$ and $u^* = (300, 0, 300)$. First we check that they are feasible for the primal and dual problems respectively. They are both nonnegative, as required. Below we also see that they satisfy the other inequality constraints:

$$x_1^* + x_2^* = 20 \leq 20,$$
$$x_1^* + 2x_2^* = 25 \leq 35,$$
$$2x_1^* + x_2^* = 35 \leq 35,$$
$$u_1^* + u_2^* + 2u_3^* = 900 \geq 900,$$
$$u_1^* + 2u_2^* + u_3^* = 600 \geq 600.$$

Letting $\nu$ be the optimal value for the primal problem, we have $\nu \geq 900x_1^* + 600x_2^* = 16500$. Letting $\omega$ be the optimal value for the dual problem, we have $\omega \leq 20u_1^* + 35u_2^* + 35u_3^* = 16500$. As these are dual problems, we have $\omega \geq \nu$, and thus $\omega = \nu = 16500$. As the points $x^* = (15, 5)$ and $u^* = (300, 0, 300)$ are feasible for their respectively problems and attain 16500 in the objective functions, they must be optimal solutions.
7. Find the dual problem to the primal problem on slide 26 of Chpt 1.

**Solution:** On slide 22 of Chpt 1 we considered the optimisation problem

\[
\min_y \left\| c - \sum_{i=1}^{m} y_i a_i \right\|_2.
\]

This is equivalent to

\[
- \max_{y,t} \begin{pmatrix} -1 \\ 0 \end{pmatrix}^T \begin{pmatrix} t \\ y \end{pmatrix} \\
\text{s.t.} \quad \begin{pmatrix} 0 \\ c \end{pmatrix} - t \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \sum_{i=1}^{m} y_i \begin{pmatrix} 0 \\ a_i \end{pmatrix} \in \mathcal{L},
\]

where \( \mathcal{L} := \{(z_0, z) \in \mathbb{R} \times \mathbb{R}^n \mid \|z\|_2 \leq z_0\} \).

The dual to this problem is

\[
- \min_x \left\langle \begin{pmatrix} 0 \\ c \end{pmatrix}, \begin{pmatrix} x_0 \\ x \end{pmatrix} \right\rangle \\
\text{s.t.} \quad \left\langle \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_0 \\ x \end{pmatrix} \right\rangle = -1 \\
\left\langle \begin{pmatrix} 0 \\ a_i \end{pmatrix}, \begin{pmatrix} x_0 \\ x \end{pmatrix} \right\rangle = 0 \quad \text{for all } i = 1, \ldots, m, \\
\begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathcal{L}^* = \mathcal{L}.
\]

This is equivalent to:

\[
\max_x - \langle c, x \rangle \\
\text{s.t.} \quad x_0 = 1 \\
\langle a_i, x \rangle = 0 \quad \text{for all } i = 1, \ldots, m, \\
\|x\|_2 \leq x_0.
\]

Simplifying and substituting \( z = -x \), this becomes:

\[
\max_z \langle c, z \rangle \\
\text{s.t.} \quad \langle a_i, z \rangle = 0 \quad \text{for all } i = 1, \ldots, m, \\
\|z\|_2 \leq 1.
\]