Continuous Optimisation: Chpt 3 Exercises

November 30, 2015

1. Consider convex cone $K \subseteq V$ and $c, a_1, \ldots, a_m \in S^n$ and $b \in \mathbb{R}^m$:

$$\begin{align*}
\min_x & \quad \langle c, x \rangle \\
\text{s.t.} & \quad \langle a_i, x \rangle = b_i \quad \text{for all } i = 1, \ldots, m \\
& \quad x \in K,
\end{align*}$$

(P)

$$\begin{align*}
\max_y & \quad b^T y \\
\text{s.t.} & \quad c - \sum_{i=1}^m y_i a_i \in K^* \\
& \quad y \in \mathbb{R}^m.
\end{align*}$$

(D)

$$K^* := \{ z \in V \mid \langle x, z \rangle \geq 0 \text{ for all } x \in K \}.$$ 

Show that $x \in \text{Feas}(P), y \in \text{Feas}(D) \Rightarrow \langle c, x \rangle \geq b^T y$.

2. Prove that $(PS^n)^* = PSD^n$.

3. Prove that $PS^n$ is a proper cone.

4. What is the dual problem to

$$\begin{align*}
\max & \quad y_1 \\
\text{s.t.} & \quad \begin{pmatrix}
1 & y_1 & 3/5 \\
y_1 & 1 & 0 \\
3/5 & 0 & 1
\end{pmatrix} \succeq 0
\end{align*}$$

5. Formulate as a semidefinite problem, the problem of finding the minimum possible 
$c_{orr}(X_1, X_2)$ given $0.5 \leq c_{orr}(X_1, X_3) \leq 0.6$ and $-0.1 \leq c_{orr}(X_2, X_3) \leq 0$.

6. Prove that $\langle A, A \rangle = \sum_{i=1}^n \lambda_i^2$
7. For the following problem, give an upper bound and formulate a PSD problem which would give a lower bound.
\[
\begin{align*}
\text{min} & \quad 2x_1^2 + 2x_2^2 - 6x_1x_2 + 2x_2 - 4x_3 \\
\text{s.t.} & \quad 2x_1^2 + x_2^2 - 2x_1x_2 - 4x_1 = 1 \\
& \quad 2x_2 - x_3^2 = 0, \quad x \in \mathbb{R}^3.
\end{align*}
\]

8. What are the dual cones to $\mathcal{DNN}^n$, $\mathcal{SPN}^n$, $\mathcal{COP}^n$ and $\mathcal{CP}^n$?
[You may assume that they are closed.]

9. Prove that $\mathcal{CP}^n \subseteq \mathcal{DNN}^n \subseteq \mathcal{SPN}^n \subseteq \mathcal{COP}^n$.

10. Prove that $\mathcal{DNN}^n$, $\mathcal{SPN}^n$, $\mathcal{COP}^n$ and $\mathcal{CP}^n$ are proper cones.
[You may assume that they are closed.]

   Hint: If $A \subseteq B$ and $B$ is pointed then $A$ is also pointed