1. Let $f_j(x), j = 1, \ldots, k (1 \leq k \in \mathbb{N}),$ be convex functions defined on a convex set $C \subset \mathbb{R}^n$.

(a) Consider with (given) $\alpha_j \geq 0, j = 1, \ldots, k,$ the function $f(x) := \sum_{j=1}^{k} \alpha_j f_j(x).$ Show that $f$ is convex on $C.$ [3 points]

(b) Show that also $g(x) := \max_{1 \leq j \leq k} \{f_j(x)\}$ is a convex function on $C.$ [3 points]

Solution:

(a) For $x, y \in C, \lambda \in [0, 1]$ we find using convexity of the $f_j$’s and $\alpha_j \geq 0$:

$$f(\lambda x + (1-\lambda)y) = \sum_{j=1}^{k} \alpha_j f_j(\lambda x + (1-\lambda)y)$$

$f_j$ convex, $\alpha_j \geq 0 \leq \sum_{j=1}^{k} \alpha_j [\lambda f_j(x) + (1-\lambda)f_j(y)]$

$$= \lambda [\sum_{j=1}^{k} \alpha_j f_j(x)] + (1-\lambda)[\sum_{j=1}^{k} \alpha_j f_j(y)]$$

$$= \lambda f(x) + (1-\lambda)f(y)$$

(b) For $x, y \in C, \lambda \in [0, 1]$ we find using convexity of the $f_j$’s:

$$g(\lambda x + (1-\lambda)y) = \max_{1 \leq j \leq k} \{f_j(\lambda x + (1-\lambda)y)\}$$

$$\leq \max_{1 \leq j \leq k} \{\lambda f_j(x) + (1-\lambda)f_j(y)\}$$

$$\leq \lambda [\max_{1 \leq j \leq k} f_j(x)] + (1-\lambda) \max_{1 \leq j \leq k} \{f_j(y)\}$$

$$= \lambda g(x) + (1-\lambda)g(y)$$

where in the second $\leq$ we used that “max of a positive sum of functions $\leq$ positive sum of max of the functions”.
2. Consider the convex program

\[
(CO) \quad \min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad x \in \mathcal{F} := \{ x \in \mathbb{R}^n \mid g_j(x) \leq 0, \ j = 1, \ldots, m \},
\]

with convex functions \( f, g_j \in C^1(\mathbb{R}^n, \mathbb{R}) \).

Show that if \( \bar{x} \in \mathcal{F} \) satisfies the KKT-conditions (Karush-Kuhn-Tucker conditions) for \((CO)\) with a multiplier vector \( \bar{y} \geq 0 \) then \((\bar{x}, \bar{y})\) is a saddle point for the Lagrangian function \( L(x, y) \) of \((CO)\).

**Solution:** KKT-conditions means that \( \bar{x} \in \mathcal{F} \) satisfies with \( \bar{y} \geq 0 \),

\[
(\nabla_x L(\bar{x}, \bar{y}) =) \quad \nabla f(\bar{x}) + \sum_{j \in J} \bar{y}_j \nabla g_j(\bar{x}) = 0 \quad \text{with} \quad \bar{y}_j g_j(\bar{x}) = 0 \ \forall j \in J.
\]

So (by Th. 3.4) \( \bar{x} \) is a global solution of \( \min_{x \in \mathbb{R}^n} L(x, \bar{y}) \) and thus

\[
L(\bar{x}, \bar{y}) \leq L(x, \bar{y}) \ \forall x \in \mathbb{R}^n \quad (\ast)
\]

Moreover since \( \bar{x} \) is feasible, i.e., \( g_j(\bar{x}) \leq 0 \ \forall j \), and using \( \bar{y}_j g_j(\bar{x}) = 0 \) we obviously obtain for all \( y \geq 0 \):

\[
L(\bar{x}, y) = f(\bar{x}) + \sum_{j \in J} y_j g_j(\bar{x}) \leq f(\bar{x}) = f(\bar{x}) + \sum_{j \in J} \bar{y}_j g_j(\bar{x}) = L(\bar{x}, \bar{y}).
\]

Together with \((\ast)\) this shows that \((\bar{x}, \bar{y})\) is a saddle point of \( L \).
3. Consider the two problems

\[(P_1) \quad \min_{x \in \mathbb{R}^2} \quad f(x) \quad \text{s.t.} \quad g_1(x) := x_1^2 - x_2 \leq 0 \]

\[(P_2) \quad \min_{x \in \mathbb{R}^2} \quad f(x) \quad \text{s.t.} \quad g_2(x) := -x_1^2 - x_2 \leq 0 \]

both with the same objective \( f(x) = 2x_1^2 + x_2 \).

(a) Which of these programs \((P_1), (P_2)\) is a convex problem? Sketch for both problems the feasible set and the level set of \( f \) given by \( f(x) = f(0,0) \). [3 points]

(b) Determine for both programs a (the) KKT-point \( \bar{x} \) with corresponding Lagrangean multiplier \( \mu \). [3 points]

(c) Show for both problems that \( \bar{x} \) is a (local) minimizer. Is it a global minimizer? [4 points]

Solution:

(a) \( f, g_1 \) are convex (e.g., show that Hessian is pos. semidef.). But \( g_2 \) is not convex, \( \nabla^2 g_2(x) = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \) is not positive semidefinite. So \((P_1)\) is convex, \((P_2)\) is not. (Give two complete sketches).

(b) The KKT-conditions read

For \((P_1)\):

\[
\begin{pmatrix} 4x_1 \\ 1 \end{pmatrix} + \mu_1 \begin{pmatrix} 2x_1 \\ -1 \end{pmatrix} = 0 \quad \text{with unique solution} \quad \mu_1 = 1, x_1 = x_2 = 0
\]

For \((P_2)\):

\[
\begin{pmatrix} 4x_1 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} -2x_1 \\ -1 \end{pmatrix} = 0 \quad \text{with unique solution} \quad \mu_2 = 1, x_1 = x_2 = 0
\]

Note that for both \( g_1, g_2 \) must be active.

(c) Since \((P_1)\) is convex the KKT-point \( \bar{x} = 0 \) must be a global minimizer (see Th. 3.7).

Since \((P_2)\) is not convex we have to check the second order conditions (in Th. 5.9) (or we can directly argue as below): we compute

\[
C_\bar{x} = \{ d \mid \nabla f(\bar{x})^T d \leq 0, \nabla g_2(\bar{x})^T d \leq 0 \} = \{ d = (d_1, d_2) \mid d_2 = 0 \}
\]

and thus

\[
d^T \nabla^2 L(\bar{x}, \mu_2) = d^T \left( \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} + \mu_2 \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \right) d = 2d_1^2 > 0
\]

for all \( d = (d_1, 0) \in C_\bar{x} \setminus \{0\} \), i.e., \( d_1 \neq 0 \). So \( \bar{x} \) is a local minimizer. It is a global minimizer since \( g_2 \leq 0 \) or \( -x_1^2 \leq x_2 \) implies:

\[
2x_1^2 + x_2 \geq 2x_1^2 - x_1^2 \geq 0 = f(\bar{x}) \quad \forall \text{feasible } x.
\]
4. Consider the auxiliary program of the SQP-method (for solving a nonlinear program $(P)$) with some $x_k \in \mathbb{R}^n$:

$$
(Q_k) \quad \min \limits_{d \in \mathbb{R}^n} \nabla f(x_k)^T d + \frac{1}{2} d^T L_k d \quad \text{s.t.} \quad \nabla g_j(x_k)^T d + g_j(x_k) \leq 0 \quad \forall j \in J
$$

Assume $x_k$ is feasible for $(P)$, i.e., $g_j(x_k) \leq 0 \forall j \in J$, and $L_k$ is positive definite. Show that if $d_k \neq 0$ is a solution of $(Q_k)$ then $d_k$ is a descent direction for $f$, i.e., $\nabla f(x_k)^T d < 0$.

**Solution:** Since $x_k$ is feasible for $(P)$, i.e., $g_j(x_k) \leq 0, \forall j$, obviously $\overline{d} = 0$ is feasible for $(Q_k)$. Since $d_k$ is a global minimizer of $(Q_k)$ (why global?; $(Q_k)$ is convex!) we must have:

$$
\nabla f(x_k)^T d_k + \frac{1}{2} d_k^T L_k d_k \leq \nabla f(x_k)^T \overline{d} + \frac{1}{2} \overline{d}^T L_k \overline{d} = 0
$$

Positive definiteness of $L_k$ implies for $d_k \neq 0$: $\nabla f(x_k)^T d_k \leq -\frac{1}{2} d_k^T L_k d_k < 0$. 

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Exam: Continuous Optimisation 2015
Monday 4th January 2016

[3 points]
5. Let $\mathcal{K} = \{ x \in \mathbb{R}^n \mid \|x\|_2 \leq 1^T x \}$, where $1 \in \mathbb{R}^n$ is the all-ones vector, and $\| \cdot \|_2$ is the Euclidean norm.

(a) Show that $\mathcal{K}$ is a proper cone. [You may assume closure.] [4 points]

(b) Show that the vectors $1$ and $(1 - e_i)$ are in $\mathcal{K}^*$ for all $i = 1, \ldots, n$, where $e_i \in \mathbb{R}^n$ is the unit vector with the first entry equal to one and all other entries equal to zero. [2 points]

(c) Show that $\mathcal{K}^* \subseteq \mathbb{R}_+^n$. [1 point]

Solution:
(a) In order to show that $\mathcal{K}$ is a proper cone, we need to show that it is a closed convex pointed full-dimensional cone. We assume closure and will now prove the rest of the properties:

- **Convex cone:**
  Let $x, y \in \mathcal{K}$ and $\lambda_1, \lambda_2 > 0$. We have $\|x\|_2 \leq 1^T x$ and $\|y\|_2 \leq 1^T y$. Therefore, letting $z = \lambda_1 x + \lambda_2 y$, we have
    \[
    \|z\|_2 = \|\lambda_1 x + \lambda_2 y\|_2 \\
    \leq \|\lambda_1 x\|_2 + \|\lambda_2 y\|_2 \\
    = \lambda_1 \|x\|_2 + \lambda_2 \|y\|_2 \\
    \leq \lambda_1 1^T x + \lambda_2 1^T y \\
    = 1^T (\lambda_1 x + \lambda_2 y) \\
    = 1^T z.
    \]

  This implies then implies that $z \in \mathcal{K}$.

- **Pointed:**
  Suppose we have $\pm x \in \mathcal{K}$. Then $\|x\|_2 \leq 1^T x$ and $\| - x\|_2 \leq 1^T (-x)$. Therefore $2\|x\|_2 = \|x\|_2 + \|-x\|_2 \leq 1^T x - 1^T x = 0$, and thus $x = 0$.

- **Full-dimensional:**
  The vectors $e_1, \ldots, e_n \in \mathbb{R}^n$ are $n$ linearly independent vectors and for all $i$ we have $\|e_i\|_2 = 1 = 1^T e_i$.

(b) We have $\mathcal{K}^* = \{ y \in \mathbb{R}^n \mid x^T y \geq 0 \text{ for all } x \in \mathcal{K} \}$.
  For all $x \in \mathcal{K}$ we have $1^T x \geq \|x\|_2 \geq 0$, and thus $1 \in \mathcal{K}^*$.
  For all $x \in \mathcal{K}$ we have $(1 - e_i)^T x = 1^T x - e_i^T x \geq \|x\|_2 - \|e_i\|_2 \|x\|_2 = 0$, and thus $(1 - e_i) \in \mathcal{K}^*$.

(c) Consider an arbitrary $x \notin \mathbb{R}_+^n$. Then there exists $i \in \{1, \ldots, n\}$ such that $x_i < 0$. From the proof in part (a) we have that $e_i \in \mathcal{K}$ and we have $\langle e_i, x \rangle = x_i < 0$, which implies that $x \notin \mathcal{K}^*$. 

5
6. Consider three random variables $X_1, X_2, X_3$. Suppose that $\text{corr}(X_1, X_2) = 0.5$ and $\text{corr}(X_1, X_3) = -0.6$.

(a) Formulate as a semidefinite optimisation problem, the problem of finding the minimum possible $\text{corr}(X_2, X_3)$. [1 point]

(b) What is the dual problem to the problem from part (a)? [2 points]

Solution:
(a)

\[
\begin{align*}
\min & \quad y_1 \\
\text{s.t.} & \quad \begin{pmatrix}
1 & 0.5 & -0.6 \\
0.5 & 1 & y_1 \\
-0.6 & y_1 & 1
\end{pmatrix} \in \mathcal{PSD}^3.
\end{align*}
\]

(b) The problem from part (a) is equivalent to

\[
-\max \quad -y_1 \\
\text{s.t.} \quad \begin{pmatrix}
1 & 0.5 & -0.6 \\
0.5 & 1 & 0 \\
-0.6 & 0 & 1
\end{pmatrix} - y_1 \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{pmatrix} \in \mathcal{PSD}^3.
\]

The dual to this is then

\[
-\min \quad \left\langle \begin{pmatrix}
1 & 0.5 & -0.6 \\
0.5 & 1 & 0 \\
-0.6 & 0 & 1
\end{pmatrix}, X \right\rangle \\
\text{s.t.} \quad \left\langle \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{pmatrix}, X \right\rangle = -1
\]

\[X \in \mathcal{PSD}^3.\]

This is equivalent to

\[
\max \quad \left\langle -\begin{pmatrix}
1 & 0.5 & -0.6 \\
0.5 & 1 & 0 \\
-0.6 & 0 & 1
\end{pmatrix}, X \right\rangle \\
\text{s.t.} \quad \left\langle \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}, X \right\rangle = 1
\]

\[X \in \mathcal{PSD}^3.\]
7. Consider the following optimisation problem:

\[
\begin{align*}
\text{max} & \quad 4x_1x_2 - x_2^2 - 9x_1 + 4x_2 \\
\text{s.t.} & \quad 4x_1^2 + x_2^2 - 8x_1 + 4x_2 + 4 = 0 \\
& \quad \mathbf{x} \in \mathbb{R}^2_+ 
\end{align*}
\]  

(A)

(a) Give a finite lower bound to the optimal value of problem (A). [1 point]

(b) Give the standard completely positive approximation for this problem, the solution of which would provide an upper bound to the optimal value of problem (A). [3 points]

\textbf{Solution:}

(a) To get a finite upper bound we need a feasible point. To narrow down the search for such a feasible point, try setting \(x_2 = 0\). Then for \(\mathbf{x}\) to be feasible we require \(4x_1^2 - 8x_1 + 4 = 0\), or equivalently \(x_1 = 1\). Therefore the point \((1, 0)\) is feasible, giving us an upper bound of \(4*1*0 - 0^2 - 9*1 + 4*0 = -9\).

(b) Problem (A) is equivalent to

\[
\begin{align*}
\text{max} & \quad 4x_1x_2 - x_2^2 - 9x_1x_3 + 4x_2x_3 \\
\text{s.t.} & \quad 4x_1^2 + x_2^2 - 8x_1x_3 + 4x_2x_3 = -4 \\
& \quad x_3^2 = 1 \quad \mathbf{x} \in \mathbb{R}^3_+.
\end{align*}
\]  

(1)

This is in turn equivalent to

\[
\begin{align*}
\text{max} & \quad \left\langle \begin{pmatrix} 0 & 2 & -9/2 \\ 2 & -1 & 2 \\ -9/2 & 2 & 0 \end{pmatrix}, \mathbf{x}\mathbf{x}^T \right\rangle \\
\text{s.t.} & \quad \left\langle \begin{pmatrix} 4 & 0 & -4 \\ 0 & 1 & 2 \\ -4 & 2 & 0 \end{pmatrix}, \mathbf{x}\mathbf{x}^T \right\rangle = -4 \\
& \quad \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{x}\mathbf{x}^T \right\rangle = 1 \\
& \quad \mathbf{x} \in \mathbb{R}^3_+.
\end{align*}
\]  

(2)
This can then be relaxed to

$$\text{max} \left\langle \begin{pmatrix} 0 & 2 & -9/2 \\ 2 & -1 & 2 \\ -9/2 & 2 & 0 \end{pmatrix}, X \right\rangle$$

s.t.

$$\left\langle \begin{pmatrix} 4 & 0 & -4 \\ 0 & 1 & 2 \\ -4 & 2 & 0 \end{pmatrix}, X \right\rangle = -4$$

$$\left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, X \right\rangle = 1$$

$$X \in \mathcal{P}^3.$$

8. (Automatic additional points) [4 points]

A copy of the lecture-sheets may be used during the examination. Good luck!