1. Let \( f : \mathbb{R}^m \to \mathbb{R} \) be a convex function \( f(y) \) on \( \mathbb{R}^m \) and let \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \) be given.

(a) Show that the function \( g(x) := f(Ax + b) \) is a convex function of \( x \) on \( \mathbb{R}^n \). \[3\] points

(b) Suppose that \( f \) is strictly convex. Show that then \( g(x) := f(Ax + b) \) is strictly convex if and only if \( A \) has (full) rank \( n \).

Hint: Recall that \( f \) is strictly convex if for any \( y_1 \neq y_2 \), \( 0 < \lambda < 1 \) it holds: 
\[
f(\lambda y_1 + (1 - \lambda) y_2) < \lambda f(y_1) + (1 - \lambda) f(y_2).
\]

Solution:

(a) For \( x_1, x_2 \in \mathbb{R}^n, \lambda \in [0, 1] \) we find:
\[
g(\lambda x_1 + (1 - \lambda)x_2) = f(A(\lambda x_1 + (1 - \lambda)x_2) + b) \\
= f(\lambda Ax_1 + (1 - \lambda)Ax_2 + \lambda b + (1 - \lambda)b) \\
= f(\lambda(Ax_1 + b) + (1 - \lambda)(Ax_2 + b)) \\
\text{if convex} \leq \lambda f(Ax_1 + b) + (1 - \lambda)f(Ax_2 + b) \\
= \lambda g(x_1) + (1 - \lambda)g(x_2)
\]

(b) \( \Leftarrow \) rank\((A) = n \) implies: \( x_1 \neq x_2 \Rightarrow Ax_1 \neq Ax_2 \).

As in (a) for \( x_1 \neq x_2, \lambda \in (0, 1) \) we obtain:
\[
g(\lambda x_1 + (1 - \lambda)x_2) = f(\lambda(Ax_1 + b) + (1 - \lambda)(Ax_2 + b)) \\
\text{"if strict convex, } Ax_1 + b \neq Ax_2 + b\" \\
< \lambda f(Ax_1 + b) + (1 - \lambda)f(Ax_2 + b) \\
= \lambda g(x_1) + (1 - \lambda)g(x_2)
\]

\( \Rightarrow \) Assume rank\((A) < n \). Then there exist \( x_1 \neq x_2 \) with \( Ax_1 = Ax_2 \)

and for any \( \lambda \in (0, 1) \) we obtain:
\[
g(\lambda x_1 + (1 - \lambda)x_2) = f(\lambda(Ax_1 + b) + (1 - \lambda)(Ax_2 + b)) = f(Ax_1 + b) \\
\text{"}g(x_1) = g(x_2)\text{"} = g(x_1) = \lambda g(x_1) + (1 - \lambda)g(x_2).
\]

So \( g \) is not strictly convex.
2. For given $S \subset \mathbb{R}^n$ we define the convex hull $\text{conv}(S)$ by

$$\text{conv}(S) = \left\{ x = \sum_{i=1}^{m} \lambda_i x_i \mid \sum_{i=1}^{m} \lambda_i = 1; x_i \in S, \lambda_i \geq 0 \forall i; m \in \mathbb{N} \right\}$$

Show that $\text{conv}(S)$ is the smallest convex set containing $S$:

(a) Show that the set $\text{conv}(S)$ is convex with $S \subset \text{conv}(S)$. [3 points]

(b) Show that for any convex set $C$ containing $S$ we must have $\text{conv}(S) \subset C$. [3 points]

(Hint: You may use without proof any Lemma, Theorem etc. from the course)

Solution:

(a) Take $x^1, x^2 \in \text{conv}(S), \lambda \in [0, 1]$ (with $x^j = \sum_{i=1}^{m_j} \lambda^j_i x^j_i, x^j_i \in S, \sum_{i=1}^{m_j} \lambda^j_i = 1, \lambda^j_i \geq 0$ for $j = 1, 2$). Then we find:

$$\lambda x^1 + (1 - \lambda)x^2 = \sum_{i=1}^{m_1} \lambda \lambda^1_i x^1_i + \sum_{i=1}^{m_2} (1 - \lambda) \lambda^2_i x^2_i \in \text{conv}(S)$$

since $\sum_{i=1}^{m_1} \lambda \lambda^1_i x^1_i + \sum_{i=1}^{m_2} (1 - \lambda) \lambda^2_i x^2_i = 1$ and “coefficients are $\geq 0$”. Note that (trivially) $S \subset \text{conv}(S)$ holds.

(b) Let $S \subset C$ with convex $C$: Take any $x \in \text{conv}(S)$, i.e., $x = \sum_{i=1}^{m} \lambda_i x_i$ with $\lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1$ and $x_i \in S$ and thus $x_i \in C$. Since $C$ is convex by Lem.2.5 (Jensen inequality) the convex combination $x$ of points $x_i \in C$ is in $C$. So $\text{conv}(S) \subset C$.

3. Consider with $0 \neq c \in \mathbb{R}^n$ the program:

$$(P) \quad \min_{x \in \mathbb{R}^n} c^T x \quad \text{s.t.} \quad x^T x \leq 1.$$ 

(a) Show that $\bar{x} = -\frac{c}{\|c\|}$ is the minimizer of $(P)$ with minimum value $v(P) = -\|c\|$. [2 points]

(||x|| means here the Euclidian norm.)

(b) Compute the solution $\bar{y}$ of the Lagrangean dual $(D)$ of $(P)$. Show in this way that for the optimal values strong duality holds, i.e., $v(D) = v(P)$. [4 points]

Solution:

(a) Either show this “by a sketch”. Or as follows (using Schwarz inequality):

$\|x\| \leq 1$ implies: $c^T x \geq -\|c\| \|x\| \geq -\|c\|$, and “$c^T x = -\|c\|$” holds iff $x = -\frac{c}{\|c\|}$

So $\bar{x} = -\frac{c}{\|c\|}$ is the minimizer with $v(P) = c^T(-\frac{c}{\|c\|}) = -\|c\|$. 

(Alternatively find $\bar{x}$ by solving the KKT-conditions.)
(b) The dual (D) is given by

\[(D) \quad \max_{y \geq 0} \psi(y) \quad \text{where} \quad \psi(y) := \min_{x \in \mathbb{R}^n} L(x, y)\]

with Lagrangean function

\[L(x, y) = c^T x + y(x^T x - 1)\]

We find for \(y = 0\):

\[\psi(0) = -\infty.\]

for \(y > 0\): The minimizer \(x\) of \(\psi(y)\) satisfies

\[\nabla_x L(x, y) = c + 2yx = 0 \quad \text{or} \quad x = -\frac{1}{2y}c.\]

So (fill in)

\[\psi(y) = -\frac{1}{2y}c^T c + \frac{1}{4y}c^T c - y = -\frac{1}{4y}c^T c - y.\]

To find an (unconstrained) maximizer of \(\psi(y)\) for \(y > 0\) we solve

\[\psi'(y) = \frac{1}{4y^2}c^T c - 1 = 0 \quad \text{with solution} \quad \bar{y} = \frac{1}{2}\|c\|.\]

So \(v(D) = \psi(\bar{y}) = -\|c\| = v(P)\).

---

4. Consider the problem (in connection with the design of a cylindrical can with height \(h\), radius \(r\) and volume at least \(2\pi\) such that the total surface area is minimal):

\[(P) : \quad \min f(h, r) := 2\pi(r^2 + rh) \quad \text{s.t.} \quad -\pi r^2 h \leq -2\pi, \quad \text{(and} \ h > 0, r > 0)\]

(a) Compute a (the) solution \((\bar{h}, \bar{r})\) of the KKT conditions of \((P)\). Show that \((P)\) is not a convex optimization problem. [4 points]

(b) Show that the solution \((\bar{h}, \bar{r})\) in (a) is a local minimizer. Why is it the unique global solution?

*Hint: Use the sufficient optimality conditions*

**Solution:**

(a) We first note that the functions \(f(h, r) = 2\pi(r^2 + rh)\) and \(g(h, r) := -\pi r^2 h + 2\pi\) are not convex (for \(h > 0\)). For the objective function \(f\), e.g., this follows by:

\[\nabla f = 2\pi \begin{pmatrix} r \\ 2r + h \end{pmatrix}, \quad \nabla^2 f = 2\pi \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and thus:} \quad \det \nabla^2 f < 0\]

We now consider the KKT condition: \((\nabla f = -\mu \nabla g, g \leq 0, \mu \cdot g = 0)\)

So consider: \(2\pi \begin{pmatrix} r \\ 2r + h \end{pmatrix} = \mu \pi \begin{pmatrix} r^2 \\ 2r^2 \end{pmatrix} \quad (\ast)\)

Case \(\mu = 0\): leads to \(2\pi \begin{pmatrix} r \\ 2r + h \end{pmatrix} = 0\) with solution \((h, r) = (0, 0)\) which is not feasible.
Case $\mu > 0$ and thus $\pi r^2 h = 2\pi$:
The 2 equations in $(\star)$ lead to $\mu = 2/r$ and then $2(2r+h) = \frac{2}{r}2rh$ or $h = 2r$.
By using the (active) constraint we find $\pi r^2 h = 2\pi r^3 = 2\pi$ with solution $r = 1$. So the unique KKT solution is given by $(\overline{h}, \overline{r}) = (2, 1), \overline{\mu} = 2$.

(b) (We apply the second order sufficient conditions of Th. 5.9 to the nonconvex program $(P)$). So we will show (for the cone of critical directions $C(\overline{h}, \overline{r})$):

$$d^T \nabla^2 L_{h,r}(\overline{h}, \overline{r}, \overline{\mu}) d > 0 \quad \forall d \in C(\overline{h}, \overline{r}) \setminus \{0\} \quad (\star\star)$$

We compute

$$\nabla f(\overline{h}, \overline{r}) = 2\pi \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad \nabla g(\overline{h}, \overline{r}) = -\pi \begin{pmatrix} 1 \\ 4 \end{pmatrix},$$

$$\nabla^2 L(\overline{h}, \overline{r}, \overline{\mu}) = 2\pi \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} + 2(-\pi) \begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix} = -2\pi \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

and

$$C(\overline{h}, \overline{r}) = \{d \in \mathbb{R}^2 \mid \nabla f(\overline{h}, \overline{r})^T d \leq 0, \nabla g(\overline{h}, \overline{r})^T d \leq 0\}$$

$$= \{d \in \mathbb{R}^2 \mid \begin{pmatrix} 1 \\ 4 \end{pmatrix}^T d \leq 0, \quad -\begin{pmatrix} 1 \\ 4 \end{pmatrix}^T d \leq 0\}$$

$$= \{\lambda \begin{pmatrix} -4 \\ 1 \end{pmatrix} \mid \lambda \in \mathbb{R}\}$$

For $d = \lambda (-4, 1)^T \neq 0$, (i.e., $\lambda \neq 0$) we obtain (see (\star\star)):

$$\lambda (-4, 1)(-2\pi) \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \lambda \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \ldots = 2\lambda^2 \pi 6 > 0 \quad \forall \lambda \neq 0.$$  

So $(\overline{h}, \overline{r}) = (2, 1)$ is a local minimizer.

It is the unique (global) minimizer since the point is the only KKT point.

Note that since the linear independency constraint qualification holds ($\nabla g = -\pi r^2 / (2rh) \neq 0$, for $r, h > 0$) any local minimizer must be a KKT point. Also note that for feasible $\|(h, r)\| \rightarrow \infty$ also $f \rightarrow \infty$ holds. (To show the latter fact is technically “involved” and was not expected to be done.)
5. Consider the closed set

\[ K = \{ x \in \mathbb{R}^2 \mid x_1 + 2x_2 \geq 0 \text{ and } 3x_1 + x_2 \geq 0 \} \]

(a) Prove that \( K \) is a proper cone. [You may assume closure.] [5 points]

(b) Find the dual cone to \( K \). [1 point]

**Solution:**

(a) In order for a set to be a proper cone it must be a closed, convex, pointed full-dimensional cone. We will assume closure and prove the rest:

- **Convex cone:** Consider an arbitrary \( x, y \in \mathbb{R}^2 \) and \( \lambda_1, \lambda_2 > 0 \). From Theorem 1.3 of the conic optimisation part of the course, if we can show that \( \lambda_1 x + \lambda_2 y \in K \) then we are done.

We have

\[
\begin{align*}
x_1 + 2x_2 &\geq 0, & 3x_1 + x_2 &\geq 0, & \lambda_1 &> 0, \\
y_1 + 2y_2 &\geq 0, & 3y_1 + y_2 &\geq 0, & \lambda_2 &> 0.
\end{align*}
\]

This implies that

\[
\begin{align*}
(\lambda_1 x + \lambda_2 y)_1 + 2(\lambda_1 x + \lambda_2 y)_2 &= \lambda_1 (x_1 + 2x_2) + \lambda_2 (y_1 + 2y_2) \geq 0, \\
3(\lambda_1 x + \lambda_2 y)_1 + (\lambda_1 x + \lambda_2 y)_2 &= \lambda_1 (3x_1 + x_2) + \lambda_2 (3y_1 + y_2) \geq 0.
\end{align*}
\]

Therefore \( \lambda_1 x + \lambda_2 y \in K \).

- **Full-dimensional:** Using Definition 1.8, part 2 of the conic optimisation part of the course, this follows from the space being two dimensional and having two linearly independent vectors \( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in K \).

- **Pointed:** We will consider an arbitrary \( x \in \mathbb{R}^2 \) such that \( \pm x \in \mathbb{K} \). Using Definition 1.7 of the conic optimisation part of the course, if we can then show that \( x = 0 \) then we are done. We have

\[
\begin{align*}
(x)_1 + 2(x)_2 &\geq 0 \\
(-x)_1 + 2(-x)_2 &\geq 0 \\
3(x)_1 + (x)_2 &\geq 0 \\
3(-x)_1 + (-x)_2 &\geq 0
\end{align*}
\]

\[ \Rightarrow \quad x_1 + 2x_2 = 0, \quad 3x_1 + x_2 = 0. \]

Therefore

\[
\begin{align*}
x_1 &= \frac{2}{5}(3x_1 + x_2) - \frac{1}{5}(x_1 + 2x_2) = 0, \\
x_2 &= (3x_1 + x_2) - 3x_1 = 0.
\end{align*}
\]
(b) From Corollary 2.8 of the conic optimisation part of the course and the note on slide 10/31 of the first lecture in the conic optimisation part of the course we have that
\[ K^* = \text{cl conic} \left\{ \begin{pmatrix} 1 \\ 2 \\ \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} = \text{conic} \left\{ \begin{pmatrix} 1 \\ 2 \\ \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}. \]

6. We will consider bounds to the optimal value of the following problem:
\[
\begin{align*}
\min_{\mathbf{x}} & \quad 5x_1^2 - 4x_1x_2 - 2x_1 + x_2^2 + 2 \\
\text{s.t.} & \quad x_1^2 + 5x_2^2 - 4x_1x_2 - 8x_2 = 4 \\
& \quad \mathbf{x} \in \mathbb{R}^2.
\end{align*}
\] (A)

(a) Give a finite upper bound on the optimal value of problem (A). [1 point]
(b) Formulate a positive semidefinite optimisation problem to give a lower bound on the optimal value of problem (A). [2 points]
(c) Give the dual problem to the positive semidefinite optimisation problem you formulated in part (b) of this question. [1 point]

**Solution:**

(a) To find an upper bound we can use any feasible point, \( \hat{\mathbf{x}} \). If we limit our search for a feasible point such that \( \hat{x}_2 = 0 \) then we would have a feasible point if and only if \( 4 = \hat{x}_1^2 + 5 \cdot 0^2 - 4 \cdot \hat{x}_1 \cdot 0 - 8 \cdot 0 = \hat{x}_1^2 \). Therefore both \( \hat{\mathbf{x}} = (2, 0) \) and \( \hat{\mathbf{x}} = (-2, 0) \) are feasible points. We only need one point to give us an upper bound, and if we consider the feasible point \( \hat{\mathbf{x}} = (2, 0) \) then this gives us the upper bound of
\[
5\hat{x}_1^2 - 4\hat{x}_1\hat{x}_2 - 2\hat{x}_1 + \hat{x}_2^2 + 2 = 5 \cdot 2^2 - 4 \cdot 2 \cdot 0 - 2 \cdot 2 + 0^2 + 2 \\
= 20 - 0 - 4 + 0 + 2 \\
= 18
\]

(b) Problem (A) is equivalent to
\[
\begin{align*}
\min_{\mathbf{x}} & \quad 5x_1^2 - 4x_1x_2 - 2x_1x_3 + x_2^2 + 2x_3^2 \\
\text{s.t.} & \quad x_1^2 + 5x_2^2 - 4x_1x_2 - 8x_2x_3 = 4 \\
& \quad x_3^2 = 1, \quad \mathbf{x} \in \mathbb{R}^3.
\end{align*}
\]
which is in turn equivalent to

\[
\min_{x,X} \left\langle \begin{pmatrix} 5 & -2 & -1 \\ -2 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}, X \right\rangle
\]

s.t. \[
\left\langle \begin{pmatrix} 1 & -2 & 0 \\ -2 & 5 & -4 \\ 0 & -4 & 0 \end{pmatrix}, X \right\rangle = 4
\]
\[
\left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, X \right\rangle = 1,
\]

\[X = xx^T, \quad x \in \mathbb{R}^3,\]

A lower bound on this is then provided by solving the optimisation problem

\[
\min_{x,X} \left\langle \begin{pmatrix} 5 & -2 & -1 \\ -2 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}, X \right\rangle
\]

s.t. \[
\left\langle \begin{pmatrix} 1 & -2 & 0 \\ -2 & 5 & -4 \\ 0 & -4 & 0 \end{pmatrix}, X \right\rangle = 4
\]
\[
\left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, X \right\rangle = 1,
\]

\[X \in \mathcal{PSD}^3.\]

(c) Considering slide 9/20 of lecture 3 of the conic optimisation part of this course we have that the dual problem is

\[
\max_y \quad 4y_1 + y_2
\]

s.t. \[
\begin{pmatrix} 5 & -2 & -1 \\ -2 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} - y_1 \begin{pmatrix} 1 & -2 & 0 \\ -2 & 5 & -4 \\ 0 & -4 & 0 \end{pmatrix} - y_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{PSD}^3.
\]

7. (Automatic additional points) [4 points]

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A copy of the lecture-sheets may be used during the examination.
Good luck!