Continuous Optimisation: Chpt 1 Exercises

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1. Read Chpt 0 from [KRT] and do exercises 0.1, 0.2, 0.3.

   0.1. Show that Euclids problem can be formulated as the quadratic optimization problem (QO):

   $$\begin{aligned} \max_{x} \quad & \frac{Hx(b-x)}{b} \\ \text{s.t.} \quad & 0 \leq x \leq b, \quad x \in \mathbb{R}. \end{aligned}$$

   0.2. Formulate Keplers problem as a nonlinear optimization problem (NLO).

   0.3. In the above concrete mixing problem the deviation of the mixture from the targeted ideal composition is measured by the Euclidean distance of the vectors $\mathbf{z} = A\mathbf{x}$ and $\mathbf{c}$. The distance of two vectors can be measured alternatively by e.g. the $\| \bullet \|_1$ or by the $\| \bullet \|_\infty$ norms. Restate the mixing problem by using these norms and show that this way, in both cases, pure linear optimization problems can be obtained.

2. Give the infimum, global minimisers, local minimisers and strict local minimisers for the following functions over $\mathbb{R}$:

   1. $f(x) = x^2 - 2x$,
   2. $f(x) = x$,
   3. $f(x) = \exp(x)$,
   4. $f(x) = x^3 - x$,
   5. $f(x) = \exp(x^3 - x)$,
   6. $f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 1 & \text{if } 1 \leq x \leq 2, \\ x - 1 & \text{if } 2 \leq x. \end{cases}$
   7. $f(x) = \exp(-1/x^2)$.

3. Prove that the intersection of (possibly infinitely many) convex sets is itself convex, i.e. for $\{\mathcal{C}_i : i \in \mathcal{I}\}$ being a collection of convex sets $\mathcal{C}_i \subseteq \mathbb{R}^n$, show that $\bigcap_{i \in \mathcal{I}} \mathcal{C}_i$ is also convex.

4. From the definition of convex functions, prove that the following functions are convex on $\mathbb{R}^n$:

   1. $f(\mathbf{x}) = \|\mathbf{x}\|$ for any semi-norm.
   2. $f(\mathbf{x}) = \mathbf{a}^\mathsf{T}\mathbf{x} + b$ for any fixed $\mathbf{a} \in \mathbb{R}^n$, $b \in \mathbb{R}$.

5. Prove Lemmas 1.40 and 1.42 from [KRT].

6. Show that $f : \mathcal{C} \to \mathbb{R}$ is a convex function if and only if its epigraph $\{(\mathbf{x}, h) \in \mathcal{C} \times \mathbb{R} : f(\mathbf{x}) \leq h\} \subseteq \mathcal{C} \times \mathbb{R}$ is a convex set.

7. Show that if $f : \mathcal{C} \to \mathbb{R}$ is a convex function then it is also a quasiconvex function.

8. Consider a compact (i.e. closed and bounded) convex set $\mathcal{C} \subseteq \mathbb{R}^n$ with nonempty interior, and let $\mathbf{a} \in \operatorname{int}(\mathcal{C})$ and

$$f(\mathbf{x}) = \min_{\lambda,\mathbf{y}} \{\lambda : \mathbf{x} = (1-\lambda)\mathbf{a} + \lambda\mathbf{y}, \quad \mathbf{y} \in \mathcal{C}, \quad \lambda \geq 0\}$$

   Show that:

   (a) $f : \mathbb{R}^n \to [0, \infty)$;
   (b) $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \leq 1\}$;
   (c) $\operatorname{int} \mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) < 1\}$;
   (d) f is convex.

   (Provided minimisation problem feasible, you may assume the optimal is attained)

9. For $Q \in \mathcal{S}$, $\mathbf{q} \in \mathbb{R}^n$ and $c \in \mathbb{R}$ consider the quadratic function $f(\mathbf{x}) = c + \mathbf{q}^\mathsf{T}\mathbf{x} + \frac{1}{2}\mathbf{x}^\mathsf{T}Q\mathbf{x}$. Show that f is convex on $\mathbb{R}^n$ if and only if Q is positive semidefinite.
   Hint: Use Theorem 1.27

10. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) = 2x_1^4 + 12x_1x_2 + 3x_2^2.$$

   For fixed parameters $a, b \in \mathbb{R}$ also consider the functions $g, h : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = f(a, x) \qquad \text{and} \qquad h(x) = f(x, b).$$

   By considering the Hessian matrices, answer the following questions:

   1. Is g convex on $\mathbb{R}$?
   2. Is h convex on $\mathbb{R}$?
   3. Is f convex on $\mathbb{R}^2$?

11. Show that if $\mathcal{C} \subseteq \mathbb{R}^n$ is a compact convex set and $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function then the **supremum** of f over $\mathcal{C}$ is attained at an extreme point of $\mathcal{C}$.

12. Assuming the first statement in Theorem 1.27 is true, prove the statement on the equivalent condition for strict convexity, i.e. for $f : \mathcal{C} \to \mathbb{R}$ such that $\mathcal{C} \subseteq \mathbb{R}^n$ is a convex set and $f \in C^1$ we have that f is strictly convex if and only if $f(\mathbf{y}) > f(\mathbf{x}) + \nabla f(\mathbf{x})^\mathsf{T}(\mathbf{y} - \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{C}$ with $\mathbf{x} \neq \mathbf{y}$.

13. Prove that convex functions $f : \mathbb{R} \to \mathbb{R}$ are continuous.

14. Give an example of a function which is convex on $[-1, 1]$, but not continuous on $[-1, 1]$.

15. Show that $f(x) = \exp(x)$ is convex on $\mathbb{R}$, and that $\exp(x) \geq 1 + x$ for all $x \in \mathbb{R}$.
Hint: To get the inequality, consider Theorem 1.27.

16. Show that $f(x) = -\ln(x)$ is convex on $x > 0$, and that for $a_1, \ldots, a_m > 0$ we have

$$(a_1 \ldots a_m)^{1/m} \leq \frac{a_1 + \ldots + a_m}{m}.$$

Hint: To get the inequality, consider Jensen's inequality.