Continuous Optimisation: Chpt 8 Exercises

November 11, 2016

1. Show that if $\mathcal{A}_1 \subseteq \mathcal{A}_2$ then $\mathcal{A}_1^* \supseteq \mathcal{A}_2^*$.
2. Prove that for $\mathcal{A} \subseteq \mathbb{R}^n$ we have $(\operatorname{conic} \mathcal{A})^* = \mathcal{A}^*$.
3. Show that for all $\mathcal{A} \subseteq \mathbb{R}^n$ we have $\mathcal{A} \subseteq \mathcal{A}^{**}$.
4. Prove that if $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathbb{R}^n$ are cones which contain the origin then $(\mathcal{K}_1 + \mathcal{K}_2)^* = \mathcal{K}_1^* \cap \mathcal{K}_2^*$.
5. Prove that a closed convex cone $\mathcal{K}$ is full-dimensional if and only if $\mathcal{K}^*$ is pointed.
   Hint: Show the equivalent result that $\mathcal{K}$ is not full-dimensional if and only if $\mathcal{K}^*$ is not pointed, and use the 3rd condition for being full-dimensional.
6. Consider the following optimisation problem:

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^2} \quad & x_1 + x_2 \\ \text{s. t.} \quad & x_1 - 2x_2 = 2 \\ & \|\mathbf{x}\|_2 \leq 1. \end{aligned}$$

   1. Write this problem in a standard form.
   2. Does Slater's condition hold for this standard form?
   3. Show that its conic dual problem is equivalent to the following problem:

$$\max_{y \in \mathbb{R}} \left\{ 2y \; - \; \left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} - y \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\|_2 \right\}.$$

   4. Consider the points $\mathbf{x}^* = (0, -1)$ and $y^* = -1$. Show that these are optimal solutions to the primal and dual problems respectively, and give the common optimal value.