Continuous Optimisation: Chpt 9 Exercises

November 18, 2016

1. Consider convex cone $\mathcal{K} \subseteq \mathbb{V}$ and $\mathbf{c}, \mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathcal{S}^n$ and $\mathbf{b} \in \mathbb{R}^m$:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \langle \mathbf{c}, \mathbf{x} \rangle \\ \text{s. t.} \quad & \langle \mathbf{a}_i, \mathbf{x} \rangle = b_i \qquad \text{for all } i = 1, \ldots, m \\ & \mathbf{x} \in \mathcal{K}, \end{aligned} \tag{P}$$

$$\begin{aligned} \max_{\mathbf{y}} \quad & \mathbf{b}^{\mathsf{T}} \mathbf{y} \\ \text{s. t.} \quad & \mathbf{c} - \sum_{i=1}^{m} y_i \mathbf{a}_i \in \mathcal{K}^* \\ & \mathbf{y} \in \mathbb{R}^m. \end{aligned} \tag{D}$$

$$\mathcal{K}^* := \{\mathbf{z} \in \mathbb{V} \mid \langle \mathbf{x}, \mathbf{z} \rangle \geq 0 \text{ for all } \mathbf{x} \in \mathcal{K}\}.$$

   Show that $\quad \mathbf{x} \in \text{Feas}(P),\, \mathbf{y} \in \text{Feas}(D) \Rightarrow \langle \mathbf{c}, \mathbf{x} \rangle \geq \mathbf{b}^{\mathsf{T}} \mathbf{y}$.

2. Prove that $(\mathcal{PSD}^n)^* = \mathcal{PSD}^n$.

3. Prove that $\mathcal{PSD}^n$ is a proper cone.

4. What is the dual problem to

$$\begin{aligned} \max \quad & y_1 \\ \text{s. t.} \quad & \begin{pmatrix} 1 & y_1 & 3/5 \\ y_1 & 1 & 0 \\ 3/5 & 0 & 1 \end{pmatrix} \succeq 0 \end{aligned}$$

5. Formulate as a semidefinite problem, the problem of finding the minimum possible $\text{corr}(X_1, X_2)$ given $0.5 \leq \text{corr}(X_1, X_3) \leq 0.6$ and $-0.1 \leq \text{corr}(X_2, X_3) \leq 0$.

6. For the following problem, give an upper bound and formulate a PSD problem which would give a lower bound.

$$\begin{aligned} \min \quad & 2x_1^2 + 2x_2^2 - 6x_1x_2 + 2x_2 - 4x_3 \\ \text{s. t.} \quad & 2x_1^2 + x_2^2 - 2x_1x_2 - 4x_1 = 1 \\ & 2x_2 - x_3^2 = 0, \quad \mathbf{x} \in \mathbb{R}^3. \end{aligned}$$