Practice Exam 1: Continuous Optimisation

1. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a convex function and let $\mathsf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$ be given.

   (a) Show that the function $g(\mathbf{x}) := f(\mathsf{A}\mathbf{x} + \mathbf{b})$ is a convex function of $\mathbf{x}$ on $\mathbb{R}^n$. [3 points]

   (b) Suppose that f is strictly convex. Show that then $g(\mathbf{x}) := f(\mathsf{A}\mathbf{x}+\mathbf{b})$ is strictly convex if and only if A has (full) rank n. [4 points]

   Hint: Recall that f is strictly convex if for any $\mathbf{x} \neq \mathbf{y}$, $0 < \lambda < 1$ it holds: $f(\lambda\mathbf{y} + (1-\lambda)y_2) < \lambda f(y_1) + (1-\lambda)f(y_2)$.

Solution:

(a) For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\lambda \in [0,1]$ we find:

$$\begin{aligned}
g(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) &= f(A(\lambda\mathbf{x} + (1-\lambda)\mathbf{y})\mathbf{b}) \\
&= f(\lambda\mathsf{A}\mathbf{x} + (1-\lambda)\mathsf{A}\mathbf{y} + \lambda b + (1-\lambda)b) \\
&= f(\lambda(\mathsf{A}\mathbf{x} + \mathbf{b}) + (1-\lambda)(\mathsf{A}\mathbf{y} + \mathbf{b})) \\
f \text{ is convex} \quad &\leq \lambda f(\mathsf{A}\mathbf{x} + \mathbf{b}) + (1-\lambda) f(\mathsf{A}\mathbf{y} + \mathbf{b}) \\
&= \lambda g(\mathbf{x}) + (1-\lambda) g(\mathbf{y})
\end{aligned}$$

(b) "$\Leftarrow$" $\operatorname{rank}(\mathsf{A}) = n$ implies: $\mathbf{x} \neq \mathbf{y} \quad \Rightarrow \quad \mathsf{A}\mathbf{x} \neq \mathsf{A}\mathbf{y}$.
As in (a) for $\mathbf{x} \neq \mathbf{y}$, $\lambda \in (0,1)$ we obtain:

$$\begin{aligned}
g(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) &= f(\lambda(\mathsf{A}\mathbf{x} + \mathbf{b}) + (1-\lambda)(\mathsf{A}\mathbf{y} + \mathbf{b})) \\
&\qquad \text{"}f \text{ is strict convex, } \mathsf{A}\mathbf{x} + \mathbf{b} \neq \mathsf{A}\mathbf{y} + \mathbf{b}\text{"} \\
&< \lambda f(\mathsf{A}\mathbf{x} + \mathbf{b}) + (1-\lambda) f(\mathsf{A}\mathbf{y} + \mathbf{b}) \\
&= \lambda g(\mathbf{x}) + (1-\lambda) g(\mathbf{y})
\end{aligned}$$

"$\Rightarrow$" Assume $\operatorname{rank}(\mathsf{A}) < n$. Then there exist $\mathbf{x} \neq \mathbf{y}$ with $\mathsf{A}\mathbf{x} = \mathsf{A}\mathbf{y}$ and for any $\lambda \in (0,1)$ we obtain:

$$\begin{aligned}
g(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) &= f(\lambda(\mathsf{A}\mathbf{x} + \mathbf{b}) + (1-\lambda)(\mathsf{A}\mathbf{y} + \mathbf{b})) = f(\mathsf{A}\mathbf{x} + \mathbf{b}) \\
\text{"}g(\mathbf{x}) = g(\mathbf{y})\text{"} \quad &= g(\mathbf{x}) = \lambda g(\mathbf{x}) + (1-\lambda) g(\mathbf{y}).
\end{aligned}$$

So g is not strictly convex.

2. For given $S \subseteq \mathbb{R}^n$ we define the convex hull $\operatorname{conv}(S)$ by

$$\operatorname{conv}(S) = \left\{ \mathbf{x} = \sum_{i=1}^m \lambda_i \mathbf{x}_i \ : \ m \in \mathbb{N}, \ \sum_{i=1}^m \lambda_i = 1; \ \ \mathbf{x}_i \in S, \ \lambda_i \geq 0 \ \forall i \right\}$$

   Show that $\operatorname{conv}(S)$ is the smallest convex set containing S:

   (a) Show that the set $\operatorname{conv}(S)$ is convex with $S \subseteq \operatorname{conv}(S)$. [3 points]

(b) Show that for any convex set C containing S we must have $\text{conv}(S) \subseteq C$. [3 points]
(*Hint: You may use without proof any Lemma/Theorem/Corollary from the course, except for Theorem 1.9.*)

Solution:

(a) For all $\mathbf{x} \in S$, letting $m = 1$ and $\lambda_1 = 1$ we see that $\mathbf{x} \in \text{conv}(S)$, and thus $S \subseteq \text{conv}(S)$.

To prove convexity we consider arbitrary $\mathbf{x}, \mathbf{y} \in \text{conv}(S)$, $\lambda \in [0, 1]$.

Then there exists $p, q \in \mathbb{N}$ and $\mathbf{x}_1, \ldots, \mathbf{x}_p \in S$ and $\mathbf{y}_1, \ldots, \mathbf{y}_q \in S$ and $\boldsymbol{\theta} \in \mathbb{R}^p_+$ and $\boldsymbol{\psi} \in \mathbb{R}^q_+$ such that

$$\mathbf{x} = \sum_{i=1}^{p} \theta_i \mathbf{x}_i, \qquad 1 = \sum_{i=1}^{p} \theta_i, \qquad \mathbf{y} = \sum_{i=1}^{q} \psi_i \mathbf{y}_i, \qquad 1 = \sum_{i=1}^{q} \psi_i.$$

We then have

$$\lambda \mathbf{x} + (1-\lambda)\mathbf{y} = \sum_{i=1}^{p} \lambda \theta_i \mathbf{x}_i + \sum_{i=1}^{q} (1-\lambda)\psi_i \mathbf{y}_i,$$
$$\lambda \theta_i \geq 0 \qquad \text{for all } i = 1, \ldots, p,$$
$$(1-\lambda)\psi_i \geq 0 \qquad \text{for all } i = 1, \ldots, q,$$
$$\sum_{i=1}^{p} \lambda \theta_i + \sum_{i=1}^{q} (1-\lambda)\psi_i = \lambda \sum_{i=1}^{p} \theta_i + (1-\lambda) \sum_{i=1}^{q} \psi_i = \lambda + (1-\lambda) = 1.$$

Therefore $\lambda \mathbf{x} + (1-\lambda)\mathbf{y} \in \text{conv}(S)$, and as $\mathbf{x}, \mathbf{y} \in \text{conv}(S)$, $\lambda \in [0, 1]$ were arbitrary, this implies that $\text{conv}(S)$ is a convex set.

(b) Consider an arbitrary convex set $C \subseteq \mathbb{R}^n$ such that $S \subseteq C$.

For $m \in \mathbb{N}$ let

$$S^m = \left\{ \mathbf{x} = \sum_{i=1}^{m} \lambda_i \mathbf{x}_i \; : \; \sum_{i=1}^{m} \lambda_i = 1; \;\; \mathbf{x}_i \in S, \; \lambda_i > 0 \; \forall i \right\}.$$

We then have $\text{conv}(S) = \bigcup_{m=1}^{\infty} S^m$.

We shall show by induction that $S^m \subseteq C$ for all $m \geq 1$, and thus $\text{conv}(S) = \bigcup_{m=1}^{\infty} S^m \subseteq C$.

For $m = 1$ we have $S^1 = S \subseteq C$.

Now suppose for the sake of induction that $S^m \subseteq C$ and consider an arbitrary $\mathbf{x} \in S^{m+1}$.

We have $\mathbf{x} = \sum_{i=1}^{m+1} \lambda_i \mathbf{x}_i$ for some $\mathbf{x}_i \in S$, $\lambda_i > 0$ for all i and $\sum_{i=1}^{m} \lambda_i = 1$.

Letting $\boldsymbol{\theta} \in \mathbb{R}^m_+$ such that $\theta_i = \lambda_i/(1 - \lambda_{m+1})$ and letting $\mathbf{y} = \sum_{i=1}^{m} \theta_i \mathbf{x}_i$ we have $\sum_{i=1}^{m} \theta_i = 1$ and thus $\mathbf{y} \in S^m \subseteq C$ by the induction hypothesis.

We then have $\mathbf{y} \in C$ and $\mathbf{x}_{m+1} \in S \subseteq C$ and $\lambda_{m+1} \in [0, 1]$ and $\mathbf{x} = (1 - \lambda_{m+1})\mathbf{y} + \lambda_{m+1}\mathbf{x}_{m+1}$. As C is a convex set, this implies that $\mathbf{x} \in C$.

3. Consider with $0 \neq \mathbf{c} \in \mathbb{R}^n$ the program:

$$(P) \qquad \min_{\mathbf{x} \in \mathbb{R}^n} \mathbf{c}^\mathsf{T}\mathbf{x} \quad \text{s.t.} \quad \mathbf{x}^\mathsf{T}\mathbf{x} \leq 1 \,.$$

   (a) Show that $\overline{\mathbf{x}} = -\frac{\mathbf{c}}{\|\mathbf{c}\|}$ is the minimizer of (P) with minimum value $v(P) = -\|\mathbf{c}\|$. ($\|\mathbf{x}\|$ means here the Euclidian norm.) [2 points]

   (b) Compute the solution $\overline{\mathbf{y}}$ of the Lagrangean dual (D) of (P). Show in this way that for the optimal values strong duality holds, i.e., $v(D) = v(P)$. [4 points]

Solution:

(a) Either show this "by a sketch". Or as follows (using Schwarz inequality): $\|\mathbf{x}\| \leq 1$ implies: $\mathbf{c}^\mathsf{T}\mathbf{x} \geq -\|\mathbf{c}\|\|\mathbf{x}\| \geq -\|\mathbf{c}\|$, and " $\mathbf{c}^\mathsf{T}\mathbf{x} = -\|\mathbf{c}\|$" holds iff $\mathbf{x} = -\frac{\mathbf{c}}{\|\mathbf{c}\|}$

So $\overline{\mathbf{x}} = -\frac{\mathbf{c}}{\|\mathbf{c}\|}$ is the minimizer with $v(P) = \mathbf{c}^\mathsf{T}(-\frac{\mathbf{c}}{\|\mathbf{c}\|}) = -\|\mathbf{c}\|$.
(Alternatively find $\overline{\mathbf{x}}$ by solving the KKT-conditions.)

(b) The dual (D) is given by

$$(D) \quad \max_{y \geq 0} \; \psi(y) \quad \text{where } \psi(y) := \min_{\mathbf{x} \in \mathbb{R}^n} L(\mathbf{x}, y)$$

with Lagrangean function $L(\mathbf{x}, y) = \mathbf{c}^\mathsf{T}\mathbf{x} + y(\mathbf{x}^\mathsf{T}\mathbf{x} - 1)$.
We find for $y = 0$: $\psi(0) = -\infty$.
<u>for $y > 0$:</u> The minimizer $\mathbf{x}$ of $\psi(y)$ satisfies $\nabla_\mathbf{x} L(\mathbf{x}, y) = \mathbf{c} + 2y\mathbf{x} = 0$ or $\mathbf{x} = -\frac{1}{2y}\mathbf{c}$. So (fill in)

$$\psi(y) = -\frac{1}{2y}\mathbf{c}^\mathsf{T}\mathbf{c} + \frac{1}{4y}\mathbf{c}^\mathsf{T}\mathbf{c} - y = -\frac{1}{4y}\mathbf{c}^\mathsf{T}\mathbf{c} - y \,.$$

The Lagrangian dual problem is thus

$$\max_{y > 0} \left\{ -\frac{1}{4y}\mathbf{c}^\mathsf{T}\mathbf{c} - y \right\}.$$

To find an (unconstrained) maximizer of $\psi(y)$ for $y > 0$ we solve

$$0 = \psi'(y) = \frac{1}{4y^2}\mathbf{c}^\mathsf{T}\mathbf{c} - 1 \quad \text{with solution} \quad \overline{\mathbf{y}} = \frac{1}{2}\|\mathbf{c}\| \,.$$

So $v(D) = \psi(\overline{\mathbf{y}}) = -\|\mathbf{c}\| = v(P)$.

4. Consider the problem (in connection with the design of a cylindrical can with height h, radius r and volume at least 2π such that the total surface area is minimal):

$$(P): \quad \min \; f(h, r) := 2\pi(r^2 + rh) \quad \text{s.t.} \; -\pi r^2 h \leq -2\pi, \quad (\text{and } h > 0, r > 0)$$

(a) Compute a (the) solution $(\overline{h}, \overline{r})$ of the KKT conditions of (P). Show that (P) is not a convex optimization problem. [4 points]

(b) Show that the solution $(\overline{h}, \overline{r})$ in (a) is a local minimizer. Why is it the unique global solution? [3 points]
Hint: Use the sufficient optimality conditions

Solution:

(a) We first note that the functions $f(h,r) = 2\pi(r^2 + rh)$ and $g(h,r) := -\pi r^2 h + 2\pi$ are not convex (for $h > 0$). For the objective function f, e.g., this follows by:

$$\nabla f = 2\pi \begin{pmatrix} r \\ 2r+h \end{pmatrix}, \qquad \nabla^2 f = 2\pi \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

By Lemma 1.21 we thus have that $\nabla^2 f$ is not positive semidefinite, thus f is not a convex function and we do not have a convex optimisation problem.

We now consider the KKT condition: $(\nabla f = -\mu \nabla g, g \le 0, \mu \cdot g = 0)$

So consider: $2\pi\binom{r}{2r+h} = \mu\pi\binom{r^2}{2rh}$ $(\star)$:

Case $\mu = 0$: leads to $2\pi\binom{r}{2r+h} = 0$ with solution $(h,r) = (0,0)$ which is not feasible.

Case $\mu > 0$ and thus $g = 0$, or equivalently $\pi r^2 h = 2\pi$:
The 2 equations in $(\star)$ lead to $\mu = 2/r$ and then $2(2r+h) = \frac{2}{r}2rh$ or $h = 2r$. By using the (active) constraint we find $\pi r^2 h = 2\pi r^3 = 2\pi$ with solution $r = 1$. So the unique KKT solution is given by $(\overline{h}, \overline{r}) = (2,1), \overline{\mu} = 2$.

(b) (We apply the second order sufficient conditions of Th. 5.14 to the nonconvex program (P)). So we will show (for the set $C = \{\mathbf{d} \in \mathbb{R}^n : \nabla f^\mathsf{T}\mathbf{d} \le 0,\ \nabla g^\mathsf{T}\mathbf{d} \le 0\}$):

$$\mathbf{d}^\mathsf{T} \nabla^2_{h,r} L(\overline{h}, \overline{r}, \overline{\mu})\mathbf{d} > 0 \quad \forall \mathbf{d} \in C \setminus \{0\} \quad (\star\star)$$

We compute

$$\nabla f(\overline{h}, \overline{r}) = 2\pi \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \qquad \nabla g(\overline{h}, \overline{r}) = -\pi \begin{pmatrix} 1 \\ 4 \end{pmatrix},$$

$$\nabla^2 L(\overline{h}, \overline{r}, \overline{\mu}) = 2\pi \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} + 2(-\pi)\begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix} = -2\pi \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

and

$$\begin{aligned} C &= \{\mathbf{d} \in \mathbb{R}^2 \mid \nabla f(\overline{h}, \overline{r})^\mathsf{T}\mathbf{d} \le 0, \nabla g(\overline{h}, \overline{r})^\mathsf{T}\mathbf{d} \le 0\} \\ &= \left\{\mathbf{d} \in \mathbb{R}^2 \mid \begin{pmatrix} 1 \\ 4 \end{pmatrix}^\mathsf{T}\mathbf{d} \le 0,\ -\begin{pmatrix} 1 \\ 4 \end{pmatrix}^\mathsf{T}\mathbf{d} \le 0\right\} \\ &= \left\{\lambda \begin{pmatrix} -4 \\ 1 \end{pmatrix} \mid \lambda \in \mathbb{R}\right\} \end{aligned}$$

For $d = \lambda(-4,1)^\mathsf{T} \neq \mathbf{0}$, (i.e., $\lambda \neq 0$) we obtain (see ($\star\star$)):

$$\lambda(-4,1)(-2\pi)\begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix}\lambda\begin{pmatrix}-4\\ 1\end{pmatrix} = \ldots = 2\lambda^2\pi 6 > 0 \quad \forall \lambda \neq 0 \;.$$

So $(\overline{h},\overline{r}) = (2,1)$ is a local minimizer.
It is the unique (global) minimizer since the point is the only KKT point. Note that since the linear independency constraint qualification holds ($\nabla g = -\pi\binom{r^2}{2rh} \neq 0$, for $r,h > 0$) any local minimizer must be a KKT point. Also note that for feasible $\|(h,r)\| \to \infty$ also $f \to \infty$ holds. (*To show the latter fact is technically "involved" and was not expected to be done.*)

5. Consider the closed set

$$\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 + 2x_2 \geq 0 \text{ and } 3x_1 + x_2 \geq 0\}$$

(a) Prove that $\mathcal{K}$ is a proper cone. [You may assume closure.] [5 points]

(b) Find the dual cone to $\mathcal{K}$. [1 point]

Solution:

(a) In order for a set to be a proper cone it most be a closed, convex, pointed full-dimensional cone. We will assume closure and prove the rest:

- Convex cone: Consider an arbitrary $\mathbf{x},\mathbf{y} \in \mathcal{K}$ and $\lambda_1,\lambda_2 > 0$. From Theorem 7.2, if we can show that $\lambda_1\mathbf{x} + \lambda_2\mathbf{y} \in \mathcal{K}$ then we are done. We have

$$\begin{aligned} x_1 + 2x_2 &\geq 0, & 3x_1 + x_2 &\geq 0, & \lambda_1 &> 0,\\ y_1 + 2y_2 &\geq 0, & 3y_1 + y_2 &\geq 0, & \lambda_2 &> 0. \end{aligned}$$

  This implies that

$$\begin{aligned} (\lambda_1\mathbf{x} + \lambda_2\mathbf{y})_1 + 2(\lambda_1\mathbf{x} + \lambda_2\mathbf{y})_2 &= \lambda_1(x_1 + 2x_2) + \lambda_2(y_1 + 2y_2) \geq 0,\\ 3(\lambda_1\mathbf{x} + \lambda_2\mathbf{y})_1 + (\lambda_1\mathbf{x} + \lambda_2\mathbf{y})_2 &= \lambda_1(3x_1 + x_2) + \lambda_2(3y_1 + y_2) \geq 0. \end{aligned}$$

  Therefore $\lambda_1\mathbf{x} + \lambda_2\mathbf{y} \in \mathcal{K}$.

- Full-dimensional: Using Definition 7.8, part 2, this follows from the space being two dimensional and having two linearly independent vectors $\begin{pmatrix}1\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 1\end{pmatrix} \in \mathcal{K}$.

- Pointed: We will consider an arbitrary $\mathbf{x} \in \mathbb{R}^2$ such that $\pm\mathbf{x} \in \mathcal{K}$. Using Definition 7.7, if we can then show that $\mathbf{x} = \mathbf{0}$ then we are done. We have

$$\left.\begin{aligned}(\mathbf{x})_1 + 2(\mathbf{x})_2 \geq 0\\ (-\mathbf{x})_1 + 2(-\mathbf{x})_2 \geq 0\end{aligned}\right\} \quad\Rightarrow\quad x_1 + 2x_2 = 0,$$

$$\left.\begin{aligned}3(\mathbf{x})_1 + (\mathbf{x})_2 \geq 0\\ 3(-\mathbf{x})_1 + (-\mathbf{x})_2 \geq 0\end{aligned}\right\} \quad\Rightarrow\quad 3x_1 + x_2 = 0.$$

Therefore

$$x_1 = \tfrac{2}{5}\underbrace{(3x_1 + x_2)}_{=0} - \tfrac{1}{5}\underbrace{(x_1 + 2x_2)}_{=0} = 0, \qquad x_2 = \underbrace{(3x_1 + x_2)}_{=0} - 3\underbrace{x_1}_{=0} = 0.$$

(b) From Theorem 8.7 we have

$$\mathcal{K}^* = \mathrm{cl}\ \mathrm{conic}\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}\right\} = \mathrm{conic}\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}\right\}.$$

6. We will consider bounds to the optimal value of the following problem:

$$\begin{aligned}\min_{\mathbf{x}} \quad & 5x_1^2 - 4x_1x_2 - 2x_1 + x_2^2 + 2\\ \text{s.t.} \quad & x_1^2 + 5x_2^2 - 4x_1x_2 - 8x_2 = 4\\ & \mathbf{x} \in \mathbb{R}^2.\end{aligned} \tag{A}$$

(a) Give a finite upper bound on the optimal value of problem (A). [1 point]

(b) Formulate a positive semidefinite optimisation problem to give a lower bound on the optimal value of problem (A). [2 points]

(c) Give the dual problem to the positive semidefinite optimisation problem you formulated in part (b) of this question. [1 point]

Solution:

(a) To find an upper bound we can use any feasible point, $\widehat{\mathbf{x}}$. If we limit our search for a feasible point such that $\widehat{x}_2 = 0$ then we would have a feasible point if and only if $4 = \widehat{x}_1^2 + 5 * 0^2 - 4\widehat{x}_1 * 0 - 8 * 0 = \widehat{x}_1^2$. Therefore both $\widehat{\mathbf{x}} = (2, 0)$ and $\widehat{\mathbf{x}} = (-2, 0)$ are feasible points. We only need one point to give us an upper bound, and if we consider the feasible point $\widehat{\mathbf{x}} = (2, 0)$

then this gives us the upper bound of

$$\begin{aligned}5\widehat{x}_1^2 - 4\widehat{x}_1\widehat{x}_2 - 2\widehat{x}_1 + \widehat{x}_2^2 + 2 &= 5 * 2^2 - 4 * 2 * 0 - 2 * 2 + 0^2 + 2 \\ &= 20 - 0 - 4 + 0 + 2 \\ &= 18\end{aligned}$$

(b) Problem (A) is equivalent to

$$\begin{aligned}\min_{\mathbf{x}} \quad & \left\langle \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}, \mathbf{x}\mathbf{x}^\mathsf{T} \right\rangle - 2x_1 + 2 \\ \text{s.t.} \quad & \left\langle \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}, \mathbf{x}\mathbf{x}^\mathsf{T} \right\rangle - 8x_2 = 4 \\ & \begin{pmatrix} 1 & \mathbf{x}^\mathsf{T} \\ \mathbf{x} & \mathbf{x}\mathbf{x}^\mathsf{T} \end{pmatrix} \in \mathcal{PSD}^3 \\ & \mathbf{x} \in \mathbb{R}^3,\end{aligned}$$

A lower bound on this is then provided by solving the optimisation problem

$$\begin{aligned}\min_{\mathbf{x}, X} \quad & \left\langle \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}, \mathsf{X} \right\rangle - 2x_1 + 2 \\ \text{s.t.} \quad & \left\langle \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}, \mathsf{X} \right\rangle - 8x_2 = 4 \\ & \begin{pmatrix} 1 & \mathbf{x}^\mathsf{T} \\ \mathbf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{PSD}^3.\end{aligned}$$

(c) This previous optimisation problem is equivalent to

$$\begin{aligned}\min_{\mathbf{x}, X} \quad & \left\langle \begin{pmatrix} 2 & -1 & 0 \\ -1 & 5 & -2 \\ 0 & -2 & 1 \end{pmatrix}, \begin{pmatrix} x_0 & \mathbf{x}^\mathsf{T} \\ \mathbf{x} & \mathsf{X} \end{pmatrix} \right\rangle \\ \text{s.t.} \quad & \left\langle \begin{pmatrix} 0 & 0 & -4 \\ 0 & 1 & -2 \\ -4 & -2 & 5 \end{pmatrix}, \begin{pmatrix} x_0 & \mathbf{x}^\mathsf{T} \\ \mathbf{x} & \mathsf{X} \end{pmatrix} \right\rangle = 4 \\ & \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} x_0 & \mathbf{x}^\mathsf{T} \\ \mathbf{x} & \mathsf{X} \end{pmatrix} \right\rangle = 1, \\ & \begin{pmatrix} x_0 & \mathbf{x}^\mathsf{T} \\ \mathbf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{PSD}^3.\end{aligned}$$

We then have that the dual problem is

$$\begin{aligned}\max_{\mathbf{y}} \quad & 4y_1 + y_2 \\ \text{s.t.} \quad & \begin{pmatrix} 2 & -1 & 0 \\ -1 & 5 & -2 \\ 0 & -2 & 1 \end{pmatrix} - y_1 \begin{pmatrix} 0 & 0 & -4 \\ 0 & 1 & -2 \\ -4 & -2 & 5 \end{pmatrix} - y_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{PSD}^3.\end{aligned}$$

7. (Automatic additional points) [4 points]

Question:	1	2	3	4	5	6	7	Total
Points:	7	6	6	7	6	4	4	40

A copy of the lecture-sheets may be used during the examination.
Good luck!