Practice Exam 1: Continuous Optimisation

1. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a convex function and let $\mathsf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$ be given.

   (a) Show that the function $g(\mathbf{x}) := f(\mathsf{A}\mathbf{x} + \mathbf{b})$ is a convex function of $\mathbf{x}$ on $\mathbb{R}^n$. [3 points]

   (b) Suppose that f is strictly convex. Show that then $g(\mathbf{x}) := f(\mathsf{A}\mathbf{x}+\mathbf{b})$ is strictly convex if and only if A has (full) rank n. [4 points]

   Hint: Recall that f is strictly convex if for any $\mathbf{x} \neq \mathbf{y}$, $0 < \lambda < 1$ it holds: $f(\lambda \mathbf{y} + (1-\lambda)y_2) < \lambda f(y_1) + (1-\lambda) f(y_2)$.

2. For given $S \subseteq \mathbb{R}^n$ we define the convex hull $\operatorname{conv}(S)$ by

$$\operatorname{conv}(S) = \left\{ \mathbf{x} = \sum_{i=1}^{m} \lambda_i \mathbf{x}_i \;:\; m \in \mathbb{N},\ \sum_{i=1}^{m} \lambda_i = 1;\ \ \mathbf{x}_i \in S,\ \lambda_i \geq 0\ \forall i \right\}$$

   Show that $\operatorname{conv}(S)$ is the smallest convex set containing S:

   (a) Show that the set $\operatorname{conv}(S)$ is convex with $S \subseteq \operatorname{conv}(S)$. [3 points]

   (b) Show that for any convex set C containing S we must have $\operatorname{conv}(S) \subseteq C$. [3 points]
   (*Hint: You may use without proof any Lemma/Theorem/Corollary from the course, except for Theorem 1.9.*)

3. Consider with $0 \neq \mathbf{c} \in \mathbb{R}^n$ the program:

$$(P) \qquad \min_{\mathbf{x}\in\mathbb{R}^n} \mathbf{c}^\mathsf{T}\mathbf{x} \quad \text{s.t.} \quad \mathbf{x}^\mathsf{T}\mathbf{x} \leq 1 \,.$$

   (a) Show that $\overline{\mathbf{x}} = -\frac{\mathbf{c}}{\|\mathbf{c}\|}$ is the minimizer of (P) with minimum value $v(P) = -\|\mathbf{c}\|$. [2 points]
   ($\|\mathbf{x}\|$ means here the Euclidian norm.)

   (b) Compute the solution $\overline{\mathbf{y}}$ of the Lagrangean dual (D) of (P). Show in this way that for the optimal values strong duality holds, i.e., $v(D) = v(P)$. [4 points]

4. Consider the problem (in connection with the design of a cylindrical can with height h, radius r and volume at least 2π such that the total surface area is minimal):

$$(P): \quad \min\ f(h,r) := 2\pi(r^2 + rh) \quad \text{s.t.} \ -\pi r^2 h \leq -2\pi, \quad (\text{and } h > 0, r > 0)$$

   (a) Compute a (the) solution $(\overline{h}, \overline{r})$ of the KKT conditions of (P). Show that (P) is not a convex optimization problem. [4 points]

   (b) Show that the solution $(\overline{h}, \overline{r})$ in (a) is a local minimizer. Why is it the unique global solution? [3 points]
   Hint: Use the sufficient optimality conditions

5. Consider the closed set

$$\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 + 2x_2 \geq 0 \text{ and } 3x_1 + x_2 \geq 0\}$$

   (a) Prove that $\mathcal{K}$ is a proper cone. [You may assume closure.] [5 points]

   (b) Find the dual cone to $\mathcal{K}$. [1 point]

6. We will consider bounds to the optimal value of the following problem:

$$\begin{aligned} \min_{\mathbf{x}} \quad & 5x_1^2 - 4x_1x_2 - 2x_1 + x_2^2 + 2 \\ \text{s.t.} \quad & x_1^2 + 5x_2^2 - 4x_1x_2 - 8x_2 = 4 \\ & \mathbf{x} \in \mathbb{R}^2. \end{aligned} \tag{A}$$

   (a) Give a finite upper bound on the optimal value of problem (A). [1 point]

   (b) Formulate a positive semidefinite optimisation problem to give a lower bound on the optimal value of problem (A). [2 points]

   (c) Give the dual problem to the positive semidefinite optimisation problem you formulated in part (b) of this question. [1 point]

7. (Automatic additional points) [4 points]

Question:	1	2	3	4	5	6	7	Total
Points:	7	6	6	7	6	4	4	40

A copy of the lecture-sheets may be used during the examination.
Good luck!