

Exam: Continuous Optimisation 2015

1. Let $f : \mathcal{C} \rightarrow \mathbb{R}$, $\mathcal{C} \subset \mathbb{R}^n$ convex, be a convex function. Show that then the following holds: [3 points]

A local minimizer of f on \mathcal{C} is a global minimizer on \mathcal{C} . And a strict local minimizer of f on \mathcal{C} is a strict global minimizer on \mathcal{C} .

2. (a) Show that for $\mathbf{d} \in \mathbb{R}^n$ it holds: [2 points]

$$\mathbf{d}^\top \mathbf{x} \geq 0 \quad \forall \mathbf{x} \in \mathbb{R}^n \quad \Leftrightarrow \quad \mathbf{d} = \mathbf{0}.$$

- (b) Let $\mathbf{c}, \mathbf{a}_i \in \mathbb{R}^n, i = 1, \dots, m$ ($m \geq 1$). Show using the Farkas Lemma (lecture sheets, Th. 5.1) that precisely one of the following alternatives (I) or (II) is true: [3 points]

(I): $\mathbf{c}^\top \mathbf{x} < 0, \mathbf{a}_i^\top \mathbf{x} \leq 0, i = 1, \dots, m$ has a solution $\mathbf{x} \in \mathbb{R}^n$.

(II): there exist $\mu_1 \geq 0, \dots, \mu_m \geq 0$ such that: $\mathbf{c} + \sum_{i=1}^m \mu_i \mathbf{a}_i = \mathbf{0}$

3. Given is the problem

$$(P) \quad \min_{\mathbf{x} \in \mathbb{R}^2} (-2x_1 - x_2) \quad \text{s.t.} \quad x_1 \leq 0, \text{ and } -(x_1 - 1)^2 - (x_2 - 1)^2 + 2 \leq 0.$$

- (a) Is (P) a convex problem? Sketch the feasible set and the level set of f given by $f(\mathbf{x}) = f(\bar{\mathbf{x}})$ with $\bar{\mathbf{x}} = \mathbf{0}$. Is LICQ (constraint qualification) satisfied at $\bar{\mathbf{x}}$? [3 points]

- (b) Show that the point $\bar{\mathbf{x}} = \mathbf{0}$ is a KKT-point of (P). Determine the corresponding Lagrangean multipliers. [3 points]

- (c) Show that $\bar{\mathbf{x}}$ is a local minimizer. What is the order of this minimizer? Is it a global minimizer? [3 points]

- (d) Consider now the program (objective f and constraint function g_2 interchanged): [2 points]

$$(\tilde{P}) \quad \min_{\mathbf{x} \in \mathbb{R}^2} -(x_1 - 1)^2 - (x_2 - 1)^2 + 2 \quad \text{s.t.} \quad x_1 \leq 0, \text{ and } -2x_1 - x_2 \leq 0.$$

Explain (without any further calculations) why $\bar{\mathbf{x}} = \mathbf{0}$ is also a local minimizer of (\tilde{P}) .

4. Consider the (nonlinear) program: [3 points]

$$(P) \quad \min_{\mathbf{x}} f(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{x} \in \mathcal{F} := \{\mathbf{x} \in \mathbb{R}^n \mid g_j(\mathbf{x}) \leq 0, j \in J\}$$

with $f, g_j \in C^1, f, g : \mathbb{R}^n \rightarrow \mathbb{R}, J = \{1, \dots, m\}$. Let \mathbf{d}_k be a strictly feasible descent direction for $\mathbf{x}_k \in \mathcal{F}$. Show that for $t > 0$, small enough, it holds:

$$f(\mathbf{x}_k + t\mathbf{d}_k) < f(\mathbf{x}_k) \quad \text{and} \quad \mathbf{x}_k + t\mathbf{d}_k \in \mathcal{F}$$

5. For a given nonempty set $\mathcal{A} \subseteq \mathbb{R}^n$ we define its conic hull, $\text{conic}(\mathcal{A})$ by

$$\text{conic}(\mathcal{A}) := \left\{ \sum_{i=1}^m \mu^i \mathbf{x}^i : \mathbf{x}^i \in \mathcal{A}, \mu^i \geq 0 \text{ for all } i, m \in \mathbb{N} \right\}.$$

- (a) Show that $\text{conic}(\mathcal{A})$ is a convex cone. [2 points]
- (b) Show that if $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathbb{R}^n$, with \mathcal{B} being a convex cone, then $\text{conic}(\mathcal{A}) \subseteq \mathcal{B}$. [3 points]
- (c) Show that $\text{conic}(\mathcal{A})$ is full dimensional if and only if there does not exist $\mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\langle \mathbf{y}, \mathbf{x} \rangle = 0$ for all $\mathbf{x} \in \mathcal{A}$. [1 point]

6. In this question we will consider the proper cone $\mathcal{K} \subseteq \mathbb{R}^{n+2}$ defined as

$$\mathcal{K} = \left\{ \begin{pmatrix} x \\ \mathbf{y} \\ z \end{pmatrix} : \mathbf{y} \in \mathbb{R}^n, x, z \in \mathbb{R}, \|\mathbf{y}\|_2 \leq x, z \geq 0 \right\}.$$

- (a) Consider a ray $\mathcal{R} = \{\mathbf{c} - y_1 \mathbf{a} \mid y_1 \in \mathbb{R}_+\}$ with fixed $\mathbf{a}, \mathbf{c} \in \mathbb{R}^n$. We wish to find the distance between the origin and the closest point in this ray. Formulate this problem as a conic optimisation problem over \mathcal{K} . [2 points]
- (b) Give an explicit characterisation of \mathcal{K}^* . [1 point]
[Justification for your answer must be provided]
- (c) What is the dual problem to your formulation in part (a)? [2 points]
[If you were not able to answer parts (a) and (b) then instead find the dual to:

$$\min_y \quad y \quad \text{s. t.} \quad \mathbf{c} + y\mathbf{a} \in \mathbb{R}_+^n. \quad]$$

7. Consider the following optimisation problem: [3 points]

$$\begin{aligned} \min_{\mathbf{x}} \quad & 2x_2^2 + 5x_1x_2 - 4x_2 \\ \text{s. t.} \quad & 2x_1^2 + x_1 + 3x_2^2 - 2x_1x_2 = 3 \\ & \mathbf{x} \in \mathbb{R}^2. \end{aligned} \quad (\text{A})$$

Give the standard positive semidefinite approximation for this problem, the solution of which would provide a lower bound to the optimal value of problem (A).

8. (Automatic additional points) [4 points]

Question:	1	2	3	4	5	6	7	8	Total
Points:	3	5	11	3	6	5	3	4	40

**A copy of the lecture-sheets may be used during the examination.
Good luck!**