DYNAMICS

- Dynamical systems can be described by a state-space model, where the state is given by a vector $x$ and the excitation by a vector $u$:

$$\frac{d}{dt}x(t) = F(x(t), u(t))$$

- A linear system is a special case:

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$

- Finding the solution of a linear system is based on finding the eigenvalues of the state transition matrix $A$.

LINEARIZATION (1)

- Nonlinear systems are difficult to solve in general. Consider the systems without excitation:

$$\frac{d}{dt}x(t) = F(x(t))$$

- States $x$ for which $F(x(t)) = 0$ are called equilibrium points.

- Suppose that $x_0$ is an equilibrium point. For points $x_0 + \Delta x$ close to this point, using Taylor-series expansion:

$$\frac{d}{dt}(x_0 + \Delta x) = F(x_0) + F'(x_0)\Delta x + \text{higher order terms}$$

with $F'(x) = \left[ \frac{\partial}{\partial x_j} f_i \right]$, the Jacobian.

LINEARIZATION (2)

- We had:

$$\frac{d}{dt}(x_0 + \Delta x) = F(x_0) + F'(x_0)\Delta x + \text{higher order terms}$$

- Neglecting the higher order terms:

$$\frac{d}{dt}\Delta x = F'(x_0)\Delta x$$

which is a linearized system. The eigenvalues of the Jacobian around equilibrium points describes the local behavior.

STABILITY

- Consider an initial state $x(0)$ in the state space close to an equilibrium point $x_0$: $|x(0) - x_0| < \delta$.

  - The system is uniformly stable if there exists a $\delta$ for any given $\epsilon$ such that $|x(t) - x_0| < \epsilon$ for all $t$.

  - The system is asymptotically stable if a $\delta$ can be found such that $\lim_{t \to \infty} x(t) = x_0$.

  - The system is marginally stable if the system is uniformly stable, but not asymptotically stable.

  - The system is unstable if it is not uniformly stable.
LYAPUNOV STABILITY (1)

* Given the system for which $\frac{d}{dt} x(t) = F(x(t))$, suppose that a function $V(x)$ exists with the following properties in a neighborhood around equilibrium points $x_0$:
  + $V(x)$ is continuous and has partial derivatives with respect to all elements of $x$.
  + $V(x_0) = 0$ for equilibrium points $x_0$ and $V(x) > 0$ for $x \neq x_0$.

* The system is uniformly stable if: $\frac{d}{dt} V(x) \leq 0$ and asymptotically stable if $\frac{d}{dt} V(x) < 0$.

LYAPUNOV STABILITY (2)

* $\frac{d}{dt} V(x) < 0$, means:
  $$\frac{\partial V}{\partial x_1} \frac{dx_1}{dt} + ... + \frac{\partial V}{\partial x_n} \frac{dx_n}{dt} = \nabla V \cdot \frac{d}{dt} x < 0$$

* Because the gradient is always perpendicular to the contour, this means that $x$ moves closer to the equilibrium point.

$V(x) = C_2$

$V(x) = C_1$

$\nabla V$