Sharp Minimax Estimation of the Variance of Brownian Motion Corrupted with Gaussian Noise

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Abstract: Let W_t be a Brownian Motion and $\epsilon_{in} \overset{iid}{\sim} \mathcal{N}(0,1)$, $i = 1, \ldots, n$ independent of W_t. $\sigma, \tau > 0$ are real, unknown parameters. Suppose we observe $Y_{i,n} = \sigma W_{i/n} + \tau \epsilon_{in}$. In this paper we will establish sharp estimators for σ^2 and τ^2 in minimax sense, i.e. they attain asymptotically the minimax constant. A short and direct proof for the minimax lower bound is given. These estimators are based on a spectral decomposition of the underlying process $Y_{i,n}$ and can be computed explicitly taking $O(n \log n)$ operations. We outline how these estimators can be generalized from Brownian Motion to processes with independent increments. Further we show that the presented spectral estimators are asymptotically normal.

Key words and phrases: Asymptotic Normality, Brownian Motion, Deconvolution, Minimax, Spectral Estimators, Statistical Inverse Problems, Variance Estimation, Oracle Estimator.

1. Introduction

Suppose we observe

$$Y_{i,n} = \sigma W_{i/n} + \tau \epsilon_{in}, \tag{1.1}$$

where W_t, $t \in [0,1]$ denotes a standard Brownian motion and $\epsilon_{in} \overset{iid}{\sim} \mathcal{N}(0,1)$. W_t and ϵ_{in} are assumed to be independent processes. We can think of the observed process as a linear combination of $W_{i/n}$ and ϵ_{in}, weighted with σ and τ, respectively. In this paper we analyze estimation of σ and τ from the viewpoint of a statistical inverse problem. From this perspective the process of interest $\sigma W_{i/n}$ is additionally corrupted by noise $\tau\epsilon_{in}$ which reveals this problem as a particular deconvolution problem. In deconvolution it is often convenient to work in the spectral domain where convolution transforms to multiplication (see e.g. Mair and Ruymgaart (1996) for an early reference) and in this paper we adopt this point of view.

Model (1.1) has received much attention during the past because it is the simplest model of high frequency financial data incorporating market microstructure noise, see e.g. Barndorff-Nielsen, Hansen, Lunde and Shephard (2007), Aït-Sahalia, Mykland and Zhang (2005) or Huang, Liu and Yu (2007) for further reading and more references. The aim is to estimate the parameters σ and τ. It is well known that σ can be estimated at a $n^{-1/4}$-rate and τ at a $n^{-1/2}$-rate, see e.g. Gloter and Jacod (2001a). In fact these are the minimax rates of convergence, i.e. the best possible rates of convergence of any estimator for τ and σ, respectively (see Tsybakov (2004) for a precise definition of a minimax rate). It is well known that the Cramer-Rao lower bound is $2\tau^4 n^{-1}$ and $8\tau\sigma^3 n^{-1/2}+o(n^{-1/2})$ for estimation of τ^2 and σ^2, respectively (Gloter and Jacod (2001a), (2001b)). These authors consider the more general model

$$\begin{aligned} Y_{in} &= X_{i/n} + \tau_n \epsilon_{in}, \quad \text{for} \quad i = 1, \ldots, n, \\ X_t &= \int_0^t \sigma(\theta, s) dW_s. \end{aligned}$$

Here, $\sigma(.,.)$ is a function satisfying some smoothness and identifiability assumptions, $\theta \in \Theta$ is the unknown parameter and $\Theta \subset \mathbb{R}$ is compact. For constant τ and $\sigma(\theta, s)$ independent of s, we receive model (1.1). If τ is known (although this is not a serious restriction), the authors obtain an estimator, based on minimizing a contrast functional, that is sharp with respect to Fisher information. They even establish LAN for their model, implying that Cramer-Rao lower bounds provide also the optimal constants in minimax sense. In this paper we will give an elementary and short proof for the sharp minimax lower bounds for estimation of τ^2 and σ^2 in model (1.1) without using LAN.

The most prominent estimator for σ^2 in model (1.1) is the maximum likelihood estimator, which is asymptotically Cramer-Rao-efficient (Stein (1987) or Aït-Sahalia, Mykland and Zhang (2005)) and hence a sharp minimax estimator. This estimator, however, requires numerical maximization of the likelihood function, which can be quite involved due to a flat likelihood function apart from its maximum. Further, the likelihood function is not convex albeit unimodal.

Therefore, a good starting value for a Newton-type iteration (or any other optimization method) is of some importance. Hence, the second goal of this paper is to construct explicitly computable estimators that are minimax sharp as well. This is easy for τ^2 but not obvious for σ^2. In order to do this we will transform the problem to the spectral domain. We will split the spectrum of the covariance in an appropriate way and mimic the linear oracle estimator. The resulting estimator is explicitly computable and only depends on the precise spectral information of the covariance of the data. Hence no numerical minimization step is involved.

We believe that our spectral approach combined with the viewpoint from nonparametric regression sheds some new light on this problem and various important facts become immediately visible. For example we see that only $\sqrt{n}$ data in the transformed model can be used for efficient estimation of σ^2, immediately revealing $n^{-1/4}$ as the minimax rate, again. In Section 4 we also indicate how these estimators can be extended to more general models and that they are robust. For simplicity we restrict ourselves in this paper to model (1.1).

We briefly mention further related work on this subject. Another estimator was introduced in Aït-Sahalia, Mykland and Zhang (2005). It does not require τ known and is asymptotically sharp in model (1.1). However, for the estimator it is necessary to minimize a complicated expression in order to calculate it.

In more general models such as in Barndorff-Nielsen, Hansen, Lunde and Shephard (2007), Zhang, Mykland and Aït-Sahalia (2005) and Jacod, Li, Mykland, Podolskij and Vetter (2007) σ is a smooth function (and possibly random) and $\int \sigma_s^2 ds$ (or as in Podolskij and Vetter (2006) $\int \sigma_s^p ds$) will be estimated. In this case the asymptotic variance for constant σ can be evaluated. So far, there is no known estimator, which is efficient with respect to this case. In fact, to our knowledge not even a sharp Cramer-Rao bound is known. The best constant $8.01\tau\sigma^3$ in this context is attained by the so called $Tuckey - Hanning_\infty$ estimator (Barndorff-Nielsen, Hansen, Lunde and Stephard (2007)) but to achieve this bound requires optimal choice of a bandwidth parameter, depending on the unknown quantities σ and τ itself.

Another interesting generalization was considered by Gloter and Hoffmann (2007). These authors replaced the Brownian motion in model (1.1) by a fractional Brownian motion with unknown Hurst index H, $1/2 < H < 1$, and proved minimax rates for estimation of σ^2 under quite general assumptions on the noise term.

The paper is organized as follows. In Section 2 we present the spectral estimators and prove that they are sharp with respect to the optimal constants in minimax sense. Computational aspects will shortly be discussed in Section 3 and we briefly investigate robustness against violations of normality and indicate the extension to more general processes with independent increments in Section 4. To keep the work more readable, all technical proofs are deferred to the supplementary material [*SM*] (`http://www.stat.sinica.edu.tw/statistica`), which contains additionally various lemmas, enumerated by $A.1, A.2, \ldots$ Some further technicalities are postponed to Appendix B.

Notation: Throughout this paper we will suppress the index n and for two sequences $(a_n)_n$ and $(b_n)_n$ we use the notation $a_n \ll b_n$ if $a_n = o(b_n)$.

2. Estimators and Sharp Minimax Bounds

Let

$$K := K_n := \operatorname{Cov}\left[W_{i/n}, W_{j/n}\right]_{i,j=1,\ldots,n} = \left(\frac{i}{n} \wedge \frac{j}{n}\right)_{i,j=1,\ldots,n}.$$

Then $Y := (Y_{1,n}, \ldots, Y_{n,n})^t \sim \mathcal{N}(0, \sigma^2 K + \tau^2 I_n)$. We can write $K = D\Lambda D^t$, where

$$D := D_n := \left(\sqrt{\frac{4}{2n+1}} \sin\left(\frac{(2j-1)\, i\pi}{2n+1}\right)\right)_{i,j=1,\ldots,n} \tag{2.1}$$

is an orthogonal matrix, i.e. $D^t D = I_n$ and Λ is a diagonal matrix with diagonal elements

$$\lambda_i := \left[4n \sin^2\left(\frac{2i-1}{4n+2}\pi\right)\right]^{-1} = \frac{1}{n} \operatorname{Dir}_n^2\left(\frac{(2i-1)\,\pi}{2n+1}\right), \tag{2.2}$$

where $\operatorname{Dir}_n(x)$ denotes the Dirichlet kernel $\operatorname{Dir}_n(x) = 1/2 + \sum_{i=1}^n \cos(i\pi x)$. This can be derived similarly as in Durbin and Knott (1972) and is based on solving a second order difference equation under given boundary conditions. Let $Z =$

$(Z_1, \dots, Z_n)^t = D^t Y$. Then,

$$Z_i \overset{ind.}{\sim} \mathcal{N}\left(0, \sigma^2\lambda_i + \tau^2\right), \quad i = 1, ..., n. \tag{2.3}$$

Hence Z_i^2, $i = 1, \dots, n$ are independent as well and have a scaled χ_1^2-distribution with expectation $\mathrm{E}\left(Z_i^2\right) = \sigma^2\lambda_i + \tau^2$ and variance $\mathrm{Var}\left(Z_i^2\right) = 2\left(\sigma^2\lambda_i + \tau^2\right)^2$. We shall work with the Z_i's from now on. Moreover they form a sufficient statistic for model (1.1).

We give here a heuristic argument that from this representation the difficulty of estimating σ^2 becomes obvious. Because $\lambda_i \asymp n/i^2$ uniformly in $i = 1, \dots, n$ (for a precise statement see Lemma B.1 in [*SM*]), only the variables Z_i^2/λ_i with $i = i(n) = O(\sqrt{n})$ have asymptotically bounded variances. Here we mean by $a_n \asymp b_n$ that $a_n = O(b_n)$ and $b_n = O(a_n)$. In contrast, for estimation of τ^2 only the "last" $n - \sqrt{n}$ variables Z_i^2 can be used. This observation is at the heart of our subsequent considerations.

2.1. Estimation of τ^2

First we consider the problem of estimating τ^2. There exist many alternatives how to define an estimator for τ^2. For instance scaled quadratic variation would work. However, in order to derive a sharp estimator of σ^2 we will need some specific preliminary estimator of τ^2, which is independent of the random variables $Z_1, \dots, Z_m$ for some $0 < m < n$. This motivates to set

$$\hat{\tau}_m^2 := \frac{1}{n-m} \sum_{i=m+1}^{n} Z_i^2, \qquad 1 < m < n. \tag{2.4}$$

Theorem 1. *Assume model (1.1) holds and let $m = m(n)$ be a sequence, such that $m/\sqrt{n} \to \infty$ and $m/n \to 0$ for $n \to \infty$. Let further the estimator $\hat{\tau}_m^2$ of τ^2 be given in (2.4). Then*

(i) $\sup_{\sigma,\tau>0} \sigma^{-2} \left|\mathrm{E}\left(\hat{\tau}_m^2\right) - \tau^2\right| = o\left(n^{-1/2}\right)$,

(ii) $n^{1/2}(\hat{\tau}_m^2 - \tau^2) \xrightarrow{L} \mathcal{N}(0,\ 2\tau^4)$, *where $\mathcal{N}\left(\mu, \sigma^2\right)$ denotes a normal r.v. with expectation μ and variance σ^2,*

(iii) and for any $\epsilon > 0$,

$$\sup_{\sigma,\tau>\epsilon} (\sigma\tau)^{-4} \left|n \,\mathrm{Var}\left(\hat{\tau}_m^2\right) - 2\tau^4\right| = o(1), \sup_{\sigma,\tau>\epsilon} (\sigma\tau)^{-4} \,\mathrm{Var}\left(\hat{\tau}_m^2\right) = O\left(n^{-1}\right).$$

Proof. *(i)* Note that

$$\mathrm{E}\left(\hat{\tau}_m^2\right) = \frac{1}{n-m}\sum_{i=m+1}^{n}\left(\sigma^2\lambda_i + \tau^2\right) = \tau^2 + \sigma^2 \frac{1}{n-m}\sum_{i=m+1}^{n}\lambda_i.$$

By Lemma B.1 and the choice of m, *(i)* follows.
(iii) It holds

$$\mathrm{Var}\left(\hat{\tau}_m^2\right) = \frac{2}{(n-m)^2}\sum_{i=m+1}^{n}\left(\tau^4 + 2\tau^2\sigma^2\lambda_i + \sigma^4\lambda_i^2\right)$$

and hence

$$\begin{aligned}
&\sup_{\sigma,\tau>\epsilon}\frac{1}{\sigma^4}\left|\frac{n}{2\tau^4}\mathrm{Var}\left(\hat{\tau}_m^2\right) - 1\right| \\
&= \sup_{\sigma,\tau>\epsilon}\left|\frac{m}{\sigma^4(n-m)} + \frac{2n}{\sigma^2\tau^2(n-m)^2}\sum_{i=m+1}^{n}\lambda_i + \frac{n}{\tau^4(n-m)^2}\sum_{i=m+1}^{n}\lambda_i^2\right| = o(1).
\end{aligned}$$

The second statement in *(iii)* follows by triangle inequality.
(ii) Note that the estimator $\hat{\tau}_m^2$ can be written as

$$\hat{\tau}_m^2 = \sum_{i=m+1}^{n}\frac{\sqrt{2}(\sigma^2\lambda_i + \tau^2)}{n-m}\cdot\frac{X_i - 1}{\sqrt{2}} + \mathrm{E}\left(\hat{\tau}_m^2\right),$$

where $X_i \overset{iid}{\sim} \chi_1^2$. Set $c_i = \sqrt{2}(\sigma^2\lambda_i + \tau^2)n^{1/2}/\left(n-m\right)$ and $R_i = \left(X_i - 1\right)/\sqrt{2}$. Then R_i are iid with mean zero and unit variance and

$$n^{1/2}(\hat{\tau}^2 - \tau^2) = \sum_{i=m+1}^{n} c_i R_i + n^{1/2}(\mathrm{E}\left(\hat{\tau}^2\right) - \tau^2).$$

Due to *(i)* and *(iii)* and since $\max_{i=m+1,\ldots,n}|c_i| = c_{m+1} \to 0$ and $\sum_{i=m+1}^{n} c_i^2 \to 2\tau^4 < \infty$ for $n \to \infty$, *(ii)* follows by using the CLT under Noether condition (see Theorem C.1 in [*SM*]). □

Note that the preceeding theorem implies that

$$\begin{aligned}
&\sup_{\sigma,\tau>\epsilon}(\sigma\tau)^{-4}\left|n\,\mathrm{MSE}\left(\hat{\tau}_m^2\right) - 2\tau^4\right| \\
&\quad\leq \sup_{\sigma,\tau>\epsilon}(\sigma\tau)^{-4}\,n\,\mathrm{Bias}^2\left(\hat{\tau}_m^2\right) + \sup_{\sigma,\tau>\epsilon}(\sigma\tau)^{-4}\left|n\,\mathrm{Var}\left(\hat{\tau}_m^2\right) - 2\tau^4\right| = o(1) \qquad (2.5)
\end{aligned}$$

and similarly

$$\sup_{\sigma,\tau>\epsilon} (\sigma\tau)^{-4} \operatorname{MSE}\left(\hat{\tau}_m^2\right) = O\left(n^{-1}\right). \tag{2.6}$$

Moreover, the constant $2\tau^4$ is sharp. More precisely, we have the following theorem.

Theorem 2. *Assume model (1.1).*

(i) Then, for any estimator $\hat{\tau}^2$, and any $\sigma \geq 0$

$$\varliminf_{n\to\infty} \inf_{\hat{\tau}^2} \sup_{\tau>\epsilon} \frac{1}{2\tau^4} \operatorname{E}\left(n\left(\hat{\tau}^2 - \tau^2\right)^2\right) \geq 1, \tag{2.7}$$

(ii) and moreover for any $\epsilon > 0$,

$$\lim_n \inf_{\hat{\tau}^2} \sup_{\sigma,\tau>\epsilon} (\sigma\tau)^{-4} \left(\operatorname{E}\left(n\left(\hat{\tau}^2 - \tau^2\right)^2\right) - 2\tau^4\right) = 0.$$

(iii) Finally, for any $0 < \epsilon < c < \infty$,

$$\lim_n \inf_{\hat{\tau}^2} \sup_{\sigma,\tau>\epsilon,\ \sigma<c} \frac{1}{2\tau^4} \operatorname{E}\left(n\left(\hat{\tau}^2 - \tau^2\right)^2\right) = 1.$$

Proof. (i) We proof this by the Information Inequality Method (see Lehmann (1983), p. 266). Note that $Z_i \overset{ind.}{\sim} \mathcal{N}(0, \sigma^2\lambda_i + \tau^2)$, $i = 1, ..., n$, can be written as $Z_i = U_i + V_i$, where $U_i \sim \mathcal{N}(0, \sigma^2\lambda_i)$, $V_i \sim \mathcal{N}(0, \tau^2)$, and $\{U_i,\ V_i,\ i = 1, ..., n\}$ are mutually independent. Estimating τ^2 based on $Z_1, ..., Z_n$ is not easier than estimating τ^2 based on $V_1, ..., V_n$ since Z_i can be generated from V_i by adding random noise U_i and is thus less informative than V_i. Hence we may assume $\sigma = 0$. The Fisher information for τ^2 is then $I\left(\tau^2\right) = n/\left(2\tau^4\right)$. Assume that (i) does not hold. Then there exists an estimator $\hat{\tau}^2$ and a subsequence $\{n_k\}$ such that

$$\lim_{k\to\infty} \sup_{\tau>\epsilon} \frac{1}{2\tau^4} \operatorname{E}\left(n_k(\hat{\tau}^2 - \tau^2)^2\right) \leq (1 - 2\delta)^2$$

for some $0 < \delta < 1/2$. Hence there exists k_1 such that for all $k \geq k_1$

$$\operatorname{E}(\hat{\tau}^2 - \tau^2)^2 \leq (1 - \delta)^2\, 2\tau^4 n_k^{-1}, \quad \text{for all } \tau > \epsilon$$

and $n_k > 50/\delta^2$. For such an n_k, the Cramer-Rao information inequality yields

$$b^2\left(\tau^2\right) + \frac{\left(1+b'\left(\tau^2\right)\right)^2}{I_{n_k}\left(\tau^2\right)} \leq (1-\delta)^2\, 2\tau^4 n_k^{-1} \quad \text{for all } \tau > \epsilon,$$

where $b\left(\tau^2\right)$ denotes the bias of $\hat{\tau}^2$. This implies both

$$b^2\left(\tau^2\right) \leq 2\tau^4 n_k^{-1} \quad \text{as well as} \quad b'\left(\tau^2\right) \leq -\delta, \quad \text{for all } \tau > \epsilon. \tag{2.8}$$

Integrating the second inequality yields $b\left(\tau^2\right) \leq -\delta\left(\tau^2 - 2\epsilon^2\right) + b\left(2\epsilon^2\right)$ for $\tau^2 \geq 2\epsilon^2$. This gives a contradiction due to

$$b\left(3\epsilon^2\right) \leq -\delta\epsilon^2 + b\left(2\epsilon^2\right) \leq -\delta\epsilon^2 + \sqrt{2}\left(2\epsilon^2\right) n_k^{-1/2} < -\sqrt{2}\left(3\epsilon^2\right) n_k^{-1/2} \leq b\left(3\epsilon^2\right),$$

where we used (2.8) in the second and last inequality and $n_k > 50/\delta^2$ in the third one. Hence (i) holds.

In order to show (ii), note that by (i)

$$\begin{aligned}&\underline{\lim}_n \inf_{\hat{\tau}^2} \sup_{\sigma,\tau>\epsilon} (\sigma\tau)^{-4}\left(\mathrm{E}\left(n\left(\hat{\tau}^2-\tau^2\right)^2\right) - 2\tau^4\right)\\ &\qquad \geq 2\,(2\epsilon)^{-4}\, \underline{\lim}_n \inf_{\hat{\tau}^2} \sup_{\tau>\epsilon,\ \sigma=2\epsilon} \left(\frac{1}{2\tau^4}\mathrm{E}\left(n\left(\hat{\tau}^2-\tau^2\right)^2\right) - 1\right) \geq 0.\end{aligned}$$

On the other hand, we have due to (2.5)

$$\begin{aligned}&\overline{\lim}_n \inf_{\hat{\tau}^2} \sup_{\sigma,\tau>\epsilon} (\sigma\tau)^{-4}\left(\mathrm{E}\left(n\left(\hat{\tau}^2-\tau^2\right)^2\right) - 2\tau^4\right)\\ &\qquad \leq \lim_n \sup_{\sigma,\tau>\epsilon} (\sigma\tau)^{-4}\left(n\,\mathrm{MSE}\left(\hat{\tau}_m^2\right) - 2\tau^4\right) = 0.\end{aligned}$$

Finally, *(iii)* follows from *(ii)* and *(i)*. □

2.2. Estimation of σ^2

We now turn to the estimation of σ^2. Define the linear oracle "estimator"

$$\hat{\sigma}^2_{\text{oracle}} := C_n^{-1} \sum_{i=1}^n \frac{\lambda_i}{(\sigma^2\lambda_i + \tau^2)^2}(Z_i^2 - \tau^2), \tag{2.9}$$

where

$$C_n := \sum_{i=1}^n \frac{\lambda_i^2}{(\sigma^2\lambda_i + \tau^2)^2}. \tag{2.10}$$

It follows from Lemma A.1 that $\hat{\sigma}^2_{\text{oracle}}$ attains the risk of $2C_n^{-1} = 8\tau\sigma^3 n^{-1/2}(1+o(1))$. Note that the oracle "estimator" $\hat{\sigma}^2_{\text{oracle}}$ depends on the unknown parameters τ^2 and σ^2 and is thus not a statistical estimator.

We shall construct below a data-driven estimator of σ^2 that mimics the performance of the oracle. For $1 < k < m < n$, set

$$\bar{\sigma}^2_{k,m} = \frac{1}{k}\sum_{i=1}^{k} \lambda_i^{-1}\left(Z_i^2 - \hat{\tau}_m^2\right). \tag{2.11}$$

Then

$$\mathrm{E}\left(\bar{\sigma}^2_{k,m}\right) = \sigma^2 + \left(\tau^2 - \mathrm{E}\left(\hat{\tau}_m^2\right)\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i^{-1}, \tag{2.12}$$

$$\mathrm{Var}\left(\bar{\sigma}^2_{k,m}\right) = \frac{1}{k^2}\sum_{i=1}^{k} 2(\sigma^2 + \tau^2\lambda_i^{-1})^2 + \mathrm{Var}\left(\hat{\tau}_m^2\right)\frac{1}{k^2}\left(\sum_{i=1}^{k}\lambda_i^{-1}\right)^2. \tag{2.13}$$

The idea is to divide the observations into three parts. Using the observations $Z_1, \ldots, Z_k$ and $Z_{m+1}, \ldots, Z_n$ in order to obtain estimates $\bar{\sigma}^2_{k,m}$ of σ^2 and $\hat{\tau}_m^2$ of τ^2 and using the middle part to construct an estimator $\hat{\sigma}^2$ by plugging in $\bar{\sigma}^2_{k,m}$ and $\hat{\tau}_m^2$ in the oracle estimator of σ^2. The advantage of this procedure is that the estimates $\bar{\sigma}^2_{k,m}$ and $\hat{\tau}^2$ are independent of the observations used for estimating σ^2 in $\hat{\sigma}^2$. For $1 \le k \ll n^{1/2} \ll m \le n$, define in analogy to (2.9) the linear oracle estimator based on the observations $Z_{k+1}, \ldots, Z_m$ by

$$\tilde{\sigma}^2_{k,m} := A_n^{-1}\sum_{i=k+1}^{m}\frac{\lambda_i}{(\sigma^2\lambda_i + \tau^2)^2}(Z_i^2 - \tau^2),$$

where $A_n := A_n(k,m) := \sum_{i=k+1}^{m}\left(\sigma^2 + \tau^2\lambda_i^{-1}\right)^{-2}$. Let $\hat{\tau}_m^2$ and $\bar{\sigma}^2_{k,m}$ be given as in (2.4) and (2.11), respectively and set $\hat{A}_n := \hat{A}_n(k,m) := \sum_{i=k+1}^{m}(\bar{\sigma}^2_{k,m} + \hat{\tau}_m^2\lambda_i^{-1})^{-2}$. Then for $1 \le k \ll n^{1/2} \ll m \le n$, define the estimator of σ^2 by

$$\hat{\sigma}^2 := \hat{\sigma}^2_{k,m} := \hat{A}_n^{-1}\sum_{i=k+1}^{m}\frac{\lambda_i}{(\bar{\sigma}^2_{k,m}\lambda_i + \hat{\tau}_m^2)^2}(Z_i^2 - \hat{\tau}_m^2). \tag{2.14}$$

Theorem 3. *Let $k = [n^{1/2-b}]$ and $m = [n^{1/2+b}]$ with $0 < b < 1/20$. Let the estimator $\hat{\sigma}^2$ of σ^2 be given in (2.14). Then, for any $\epsilon > 0$*

(i) $\sup_{\sigma,\tau>\epsilon}(\sigma\tau)^{-2}\left|\mathrm{E}\left(\hat{\sigma}^2 - \sigma^2\right)\right| = o\left(n^{-1/4}\right)$,

(ii) $\sup_{\sigma,\tau>\epsilon}(\sigma\tau)^{-8}\left|\mathrm{Var}\left(\hat\sigma^2\right)-8\tau\sigma^3 n^{-1/2}\right| = o\left(n^{-1/2}\right)$,

(iii) $n^{1/4}(\hat\sigma^2-\sigma^2)\xrightarrow{L}\mathcal{N}(0,\ 8\tau\sigma^3)$.

Proof: For ease of notation, we write in the following $\tilde\sigma^2, \bar\sigma^2$ and $\hat\tau^2$ instead of $\tilde\sigma^2_{k,m}, \bar\sigma^2_{k,m}$ and $\hat\tau^2_m$, respectively. Let us introduce the oracle estimator

$$\hat\sigma^2_\tau := \hat A_n^{-1}\sum_{i=k+1}^m \frac{\lambda_i}{(\bar\sigma^2\lambda_i+\hat\tau^2)^2}(Z_i^2-\tau^2).$$

(i) By construction we have that $\bar\sigma^2$ and Z_i as well as $\hat\tau^2$ and Z_i for $i=k+1,\ldots,m$ are independent. Hence $\mathrm{E}\left(\hat\sigma^2_\tau\right)=\sigma^2$ and due to

$$\left|\hat\sigma^2-\hat\sigma^2_\tau\right| = \hat A_n^{-1}\sum_{i=k+1}^m \frac{\lambda_i}{\left(\bar\sigma^2\lambda_i+\hat\tau^2\right)^2}\left|\hat\tau^2-\tau^2\right| \le \lambda_m^{-1}\left|\hat\tau^2-\tau^2\right| \tag{2.15}$$

also

$$\sup_{\sigma,\tau>\epsilon}\frac{1}{\sigma^2\tau^2}\left|\mathrm{E}\left(\hat\sigma^2-\sigma^2\right)\right| \le \sup_{\sigma,\tau>\epsilon}\frac{1}{\sigma^2\tau^2}\mathrm{E}\left(\left|\hat\sigma^2-\hat\sigma^2_\tau\right|\right) \le \sup_{\sigma,\tau>\epsilon}\frac{\lambda_m^{-1}}{\sigma^2\tau^2}\mathrm{MSE}^{1/2}\left(\hat\tau^2\right).$$

By (2.6) this gives *(i)*.

(ii) We have the decomposition $\hat\sigma^2-\tilde\sigma^2=\left(\hat\sigma^2-\hat\sigma^2_\tau\right)+\left(\hat\sigma^2_\tau-\tilde\sigma^2\right)$. In order to show that $\hat\sigma^2$ and $\tilde\sigma^2$ have the same asymptotic variances, we will bound the variance of the differences $\hat\sigma^2-\hat\sigma^2_\tau$ and $\hat\sigma^2_\tau-\tilde\sigma^2$. Therefore, note

$$\mathrm{Var}\left(\hat\sigma^2-\tilde\sigma^2\right)\le 2\,\mathrm{Var}\left(\hat\sigma^2-\hat\sigma^2_\tau\right)+2\,\mathrm{Var}\left(\hat\sigma^2_\tau-\tilde\sigma^2\right). \tag{2.16}$$

First we see that by (2.15)

$$\mathrm{Var}\left(\hat\sigma^2-\hat\sigma^2_\tau\right)\le \mathrm{E}\left(\hat\sigma^2-\hat\sigma^2_\tau\right)^2\le\lambda_m^{-2}\,\mathrm{MSE}\left(\hat\tau^2\right). \tag{2.17}$$

Write $Z_i^2=\left(\sigma^2\lambda_i+\tau^2\right)U_i$, where $U_i\sim\chi^2_1$, i.i.d. Let $w_{in}=A_n^{-1}\lambda_i/(\sigma^2\lambda_i+\tau^2)$ and $\hat w_{in}=\hat A_n^{-1}\lambda_i\left(\sigma^2\lambda_i+\tau^2\right)/(\bar\sigma^2\lambda_i+\hat\tau^2)^2$. Then

$$\hat\sigma^2_\tau-\tilde\sigma^2=\hat\sigma^2_\tau-\sigma^2+\sigma^2-\tilde\sigma^2=\sum_{i=k+1}^m\left(\hat w_{in}-w_{in}\right)\left(U_i-1\right).$$

By construction we have that $\hat w_{in}$ and U_i, $i=k+1,\ldots,m$ are independent. Therefore $\mathrm{E}\left(\hat\sigma^2_\tau-\tilde\sigma^2\right)=0$ and because of $\mathrm{E}\left(\sum_{i=k+1}^m w_{in}\hat w_{in}\right)=A_n^{-1}$

$$\mathrm{Var}\left(\hat\sigma^2_\tau-\tilde\sigma^2\right)=2\,\mathrm{E}\left(\sum_{i=k+1}^m\left(\hat w_{in}-w_{in}\right)^2\right)=2\,\mathrm{E}\left(\sum_{i=k+1}^m\hat w_{in}^2\right)-2A_n^{-1}.$$

Furthermore, using the inequality

$$x^2 = y^2 + 2y(x-y) + (x-y)^2 \le \left(1+a^{-1}\right) y^2 + (1+a)(x-y)^2, \quad x, y \in \mathbb{R}, a > 0$$

we obtain

$$\left(\sigma^2\lambda_i + \tau^2\right)^2 \le \left(1+n^{-b}\right)\left(\bar{\sigma}^2\lambda_i + \hat{\tau}^2\right)^2 + 2\left(1+n^b\right)\left[(\sigma^2-\bar{\sigma}^2)^2\lambda_i^2 + (\tau^2-\hat{\tau}^2)^2\right]. \tag{2.18}$$

With

$$\gamma_n := \hat{A}_n^{-2} \sum_{i=k+1}^{m} \frac{\lambda_i^2}{(\bar{\sigma}^2\lambda_i + \hat{\tau}^2)^4}\left[(\sigma^2-\bar{\sigma}^2)^2\lambda_i^2 + (\tau^2-\hat{\tau}^2)^2\right]$$

it holds

$$\begin{aligned}\sum_{i=k+1}^{m} \hat{w}_{in}^2 &= \hat{A}_n^{-2} \sum_{i=k+1}^{m} \frac{\lambda_i^2}{(\bar{\sigma}^2\lambda_i + \hat{\tau}^2)^4}(\sigma^2\lambda_i + \tau^2)^2 \\ &\le \hat{A}_n^{-1}\left(1+n^{-b}\right) + 2\left(1+n^b\right)\gamma_n.\end{aligned} \tag{2.19}$$

It follows from (2.19) and Lemmas A.2 and A.3 in [*SM*] that

$$\sup_{\sigma,\tau>\epsilon} (\sigma\tau)^{-8} \operatorname{Var}\left(\hat{\sigma}_\tau^2 - \tilde{\sigma}^2\right) = o\left(n^{-1/2}\right)$$

and hence with (2.6) and (2.17) this gives for (2.16)

$$\begin{aligned}&\sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^8\tau^8} \operatorname{Var}\left(\hat{\sigma}^2 - \tilde{\sigma}^2\right) \\ &\quad \le \sup_{\sigma,\tau>\epsilon} \frac{2}{\sigma^8\tau^8}\lambda_m^{-2} \operatorname{MSE}\left(\hat{\tau}^2\right) + o\left(n^{-1/2}\right) = o\left(n^{-1/2}\right).\end{aligned} \tag{2.20}$$

Therefore *(ii)* follows by Lemma A.1 and

$$\begin{aligned}&\left|\operatorname{Var}\left(\hat{\sigma}^2\right) - 8\tau\sigma^3 n^{-1/2}\right| \\ &\quad \le \operatorname{Var}\left(\hat{\sigma}^2 - \tilde{\sigma}^2\right) + 2^{3/2}\operatorname{Var}^{1/2}\left(\hat{\sigma}^2 - \tilde{\sigma}^2\right) A_n^{-1/2} + 2\left|A_n^{-1} - 4\tau\sigma^3 n^{-1/2}\right|,\end{aligned}$$

where we used $\operatorname{Var}\left(\tilde{\sigma}^2\right) = 2A_n^{-1}$.
(iii) Since by (i) and (2.20) $\mathrm{E}\left(\hat{\sigma}^2 - \tilde{\sigma}^2\right) = \mathrm{E}\left(\hat{\sigma}^2 - \sigma^2\right) = o\left(n^{-1/4}\right)$ and $\operatorname{Var}\left(\hat{\sigma}^2 - \tilde{\sigma}^2\right) = o\left(n^{-1/2}\right)$ we have $\hat{\sigma}^2 - \tilde{\sigma}^2 = o_P(n^{-1/4})$. Therefore we can write $n^{1/4}\left(\hat{\sigma}^2 - \sigma^2\right) = n^{1/4}\left(\tilde{\sigma}^2 - \sigma^2\right) + o_P(1)$. For the asymptotic normality

we apply again the CLT under Noether condition (Theorem C.1 in [*SM*]). We write $n^{1/4}\left(\tilde{\sigma}^2-\sigma^2\right) = n^{1/4}\sum_{i=k+1}^m w_{in}\left(U_i-1\right)$. Because of $\mathrm{E}\left(\tilde{\sigma}^2\right) = \sigma^2$ and $\operatorname{Var}\left(\tilde{\sigma}^2\right) = 2A_n^{-1}$ we only need to show that $\max_{i=k+1,\dots,m} n^{1/4}w_{in} \to 0$. To see this note that

$$\max_{i=k+1,\dots,m} n^{1/4}w_{in} \leq \frac{1}{\sigma^2}n^{1/4}A_n^{-1} \to 0,$$

where we used Lemma A.1 (ii). This proves the asymptotic normality. □

The constant $8\tau\sigma^3$ is sharp. As mentioned, the sharp minimax lower bound already follows by Theorem 12.1 in Ibragimov and Has'minskii (1981) from the LAN-property proved in Gloter and Jacod (2001a). However, we will give a short and easily accessible proof which does not require the LAN property and instead of assuming σ to be in a compact set, we may allow $\sigma,\tau \in [\epsilon,\infty)$, for some $\epsilon > 0$.

Theorem 4. *(i) For any estimator $\hat{\sigma}^2$, we have*

$$\varliminf_{n\to\infty}\sup_{\tau,\sigma>\epsilon}\frac{1}{8\tau\sigma^3}\,\mathrm{E}\left(n^{1/2}(\hat{\sigma}^2-\sigma^2)^2\right) \geq 1 \tag{2.21}$$

and equality holds if in addition $\sigma,\tau \leq K < \infty$. Furthermore,

(ii)

$$\lim_{n\to\infty}\inf_{\hat{\sigma}^2}\sup_{\tau,\sigma>\epsilon}(\sigma\tau)^{-8}\left(\mathrm{E}\left(n^{1/2}(\hat{\sigma}^2-\sigma^2)^2\right)-8\tau\sigma^3\right) = 0.$$

Proof. (i) The method of proof is similar to that of Theorem 2. Note that $Z_i \overset{ind.}{\sim} \mathcal{N}(0,\sigma^2\lambda_i+\tau^2)$, $i = 1,\dots,n$. Straightforward calculations show that the Fisher information about σ^2 contained in $Z_1,\dots,Z_n$ is $I_n(\sigma^2) = 1/2C_n = 1/2\sum_{i=1}^n 1/\left(\sigma^2+\tau^2\lambda_i^{-1}\right)^2$, where C_n is as defined in (2.10). Suppose (2.21) does not hold. Then there exists an estimator $\hat{\sigma}^2$ such that for a subsequence $\{n_k\}$

$$\lim_{k\to\infty}\sup_{\tau,\sigma>\epsilon}\frac{1}{8\tau\sigma^3}\,\mathrm{E}\left(n_k^{1/2}(\hat{\sigma}^2-\sigma^2)^2\right) \leq 1-4\delta$$

for some $0 < \delta \leq 1/4$. Hence there exists k_1 such that for all $k \geq k_1$

$$\mathrm{E}\left(\hat{\sigma}^2-\sigma^2\right)^2 \leq (1-3\delta)8\tau\sigma^3 n_k^{-1/2}, \quad \text{for all } \tau,\sigma > \epsilon. \tag{2.22}$$

Let $\tau_0 > \epsilon$ be fixed. It follows from Lemma A.1 (i) that for all $\epsilon^2 < \sigma^2 \leq 3\epsilon^2$ and all sufficiently large n_k

$$I_{n_k}(\sigma^2) \leq (1+\delta)\frac{1}{8\tau_0\sigma^3}n_k^{1/2}. \tag{2.23}$$

Hence there exists an $n_0 > 0$ such that (2.22), (2.23) and

$$n_k > \frac{64\tau_0^2\epsilon^4\left(2^{3/2}+3^{3/2}\right)^4}{\delta^4} \tag{2.24}$$

hold for all $n_k > n_0$ where (2.24) will be required later on. For such an n_k, the Cramer-Rao information inequality yields

$$b^2(\sigma^2) + \frac{(1+b'(\sigma^2))^2}{I_{n_k}(\sigma^2)} \leq (1-3\delta)8\tau_0\sigma^3 n_k^{-1/2}, \quad \text{for all } \epsilon^2 < \sigma^2 \leq 3\epsilon^2,$$

where $b(\sigma^2)$ denotes the bias of $\hat{\sigma}^2$. This implies that

$$b^2(\sigma^2) \leq 8\tau_0\sigma^3 n_k^{-1/2} \quad \text{and} \quad b'(\sigma^2) \leq -\delta, \quad \text{for all } \epsilon^2 < \sigma^2 \leq 3\epsilon^2, \tag{2.25}$$

where the latter inequality follows from $\left(1+b'\left(\theta\right)\right)^2 \leq (1-3\delta)\left(1+\delta\right)$. Further, $b'(\sigma^2) \leq -\delta$ gives

$$b(\sigma^2) \leq -\delta\left(\sigma^2 - 2\epsilon^2\right) + b(2\epsilon^2) \quad \text{for } 2\epsilon^2 \leq \sigma^2 \leq 3\epsilon^2. \tag{2.26}$$

Now with $\sigma^2 = 3\epsilon^2$ in (2.26) we obtain a contradiction for $n_k > n_0$ since

$$\begin{aligned} b\left(3\epsilon^2\right) &\leq -\delta\epsilon^2 + b\left(2\epsilon^2\right) \leq -\delta\epsilon^2 + \sqrt{8\tau_0}\left(2\epsilon^2\right)^{3/2} n_k^{-1/4} \\ &< -\sqrt{8\tau_0}\left(3\epsilon^2\right)^{3/2} n_k^{-1/4} \leq b\left(3\epsilon^2\right), \end{aligned}$$

where the second and the last inequality follow from (2.25) and the third one follows from (2.24). This proves the first part of *(i)*.

(ii) can be deduced in the same way as (ii) in Theorem 2 by using Theorem 3 and (i).

Combining *(ii)* and the first part of *(i)* gives equality in (i), if $\sigma, \tau \leq K < \infty$. □

3. Computational Aspects

Finally, we will discuss the computational complexity for calculating the spectral estimator. First we stress that this estimator can be implemented easily and in a straightforward manner. Note that the transform matrices D and D^t

defined in (2.1) are discrete sine transforms (DST-IIo, DST-IIIo). For a reference see Britanak, Yip and Yao (2006) and Curci and Corsi (2006). Discrete sine transforms behave similar as Fourier transforms and fast algorithms are available. In fact, if we have n observations performing the transformation requires $O(n \log n)$ operations. Additionally, computing $\hat{\tau}^2$, $\tilde{\sigma}^2$ and finally $\hat{\sigma}^2$ in the transformed model needs $O(n)$ steps. Hence the overall complexity is $O(n \log n)$.

Alternatively to our approach one could investigate numerically the performance of maximum likelihood methods in the difference model, where we have observations $(Y_{1,n}, Y_{2,n} - Y_{1,n}, \ldots, Y_{i,n} - Y_{i-1,n}, \ldots, Y_{n,n} - Y_{n-1,n})$ as well as in the transformed model (2.3), see Aït-Sahalia, Mykland and Zhang (2005). This leads to maximum likelihood estimation of the parameters of an $AR(1)$ process. We mention that for computation of the maximum likelihood estimator a good starting value is of crucial importance due to the flat likelihood function in regions far apart from the maximum. One might use our estimator as a starting value and then iterate a few times to obtain the maximum likelihood estimator.

4. Discussion: Extension to Other Processes

Transforming the difference vector $(Y_{n,n} - Y_{n-1,n}, \ldots, Y_{2,n} - Y_{1,n}, Y_{1,n})$ by D^t as defined in (2.1) gives us again a vector with independent observations and we can follow the same arguments to obtain a sharp estimator. From the discussion so far it is not clear how this estimation method behaves if we consider more general models since, at a first glance, D^t seems to define a global transformation. However, suppose that τ and σ are sufficiently smooth functions, slight modifications of the in (2.4), (2.11) and (2.14) proposed estimators generalize to rate optimal estimators of $\int \tau_s^2 ds$ and $\int \sigma_s^2 ds$. Obviously our technique can be directly extended if we substitute the Brownian motion in model (1.1) by a centered Lévy-Process X with initial value $X_0 = 0$ a.s. and $\mathrm{E}\left(X_1^4\right) < \infty$ (for instance a compensated Poisson process), which is independent of $\epsilon_{i,n}$. As seen above D^t defines a discrete sine transform. There is a strong connection between Karhunen-Loeve expansions and sine transforms. In fact, for wide classes of processes, D^t diagonalizes them approximately, for instance for general $\mathrm{MA}(q)$ processes. This gives us reason to believe that our approach is robust against various cases of model misspecification. However, our aim was not to discuss these models in full generality rather than to lay out these ideas as simple as possible.

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Sharp Minimax Estimation of the Variance of Brownian Motion Corrupted with Gaussian Noise: Supplementary Material

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Abstract: This note provides details of proofs and supplementary technicalities for the paper "Sharp Minimax Estimation of the Variance of Brownian Motion Corrupted with Gaussian Noise".

Appendix A. Additional Lemmas for the Risk Estimation of $\hat{\sigma}^2$

Notation: We will suppress the index n and for two sequences $(a_n)_n$ and $(b_n)_n$ we use the notation $a_n \ll b_n$ if $a_n = o(b_n)$.

Lemma A.1. *Let* $A_n := A_n(k,m) := \sum_{i=k+1}^{m} (\sigma^2 + \tau^2 \lambda_i^{-1})^{-2}$ *and* $C_n := A_n(1,n)$, *where* λ_i *is as defined in (2.2). Then, for any* $\epsilon > 0$

(i)

$$\sup_{\sigma,\tau>\epsilon} \left| C_n - \frac{1}{4\tau\sigma^3} n^{1/2} \right| = o\left(n^{1/2}\right),$$
$$\sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^5\tau^5} \left| C_n^{-1} - 4\tau\sigma^3 n^{-1/2} \right| = o\left(n^{-1/2}\right), \tag{A.1}$$

(ii) and if $k \ll n^{1/2} \ll m$ *also*

$$\sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^8\tau^8} \left| A_n^{-1} - 4\tau\sigma^3 n^{-1/2} \right| = o\left(n^{-1/2}\right),$$

and

$$\sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^4\tau^4} A_n^{-1} = O\left(n^{-1/2}\right).$$

Proof. *(i)* Let us fix the notation

$$I_n := 2n\int_0^{1/2} \frac{1}{\left(\sigma^2+\tau^2 4n\sin^2(x\pi)\right)^2}dx = \frac{32n^3\tau^4\sigma^2+10n^2\tau^2\sigma^4+32n^4\tau^6+n\sigma^6}{\left(\sigma^2+4n\tau^2\right)^{7/2}\sigma^3}.$$

By Taylor expansion and monotonicity in σ^2, we have

$$\left(\sigma^2+4n\tau^2\right)^{7/2}-\left(4n\tau^2\right)^{7/2} \le 7\left(\sigma^2+4n\tau^2\right)^{5/2}\sigma^2. \tag{A.2}$$

Note that Lemma B.2 implies for $n \ge 2$

$$\begin{aligned}
\sup_{\sigma,\tau>\epsilon}\left|C_n-\frac{1}{4\tau\sigma^3}n^{1/2}\right| &\le O(\log n)+\sup_{\sigma,\tau>\epsilon}\left|I_n-\frac{1}{4\tau\sigma^3}n^{1/2}\right| \\
&= O(\log n)+\sup_{\sigma,\tau>\epsilon}\left|\frac{32n^3\tau^4\sigma^2+10n^2\tau^2\sigma^4+32n^4\tau^6+n\sigma^6}{\left(\sigma^2+4n\tau^2\right)^{7/2}\sigma^3}-\frac{1}{4\tau\sigma^3}n^{1/2}\right| \\
&\le O(\log n)+\frac{1}{\epsilon(1+4n)^{1/2}}\sup_{\sigma,\tau>\epsilon}\left|\frac{32n^3\tau^4+10n^2\tau^2\sigma^2+n\sigma^4}{\left(\sigma^2+4n\tau^2\right)^3\sigma}\right| \\
&\quad+\sup_{\sigma,\tau>\epsilon}\frac{1}{\sigma^3}\left|\frac{\left(\sigma^2+4n\tau^2\right)^{7/2}-2^7\tau^7n^4}{4\tau\left(\sigma^2+4n\tau^2\right)^{7/2}}\right| \\
&= O(\log n)+\sup_{\sigma,\tau>\epsilon}\frac{7n^{1/2}}{4\sigma\tau\left(\sigma^2+4n\tau^2\right)} = O(\log n).
\end{aligned}$$

Finally we will show (A.1). Note

$$\left|C_n^{-1}-4\tau\sigma^3n^{-1/2}\right| \le \left|C_n^{-1}-I_n^{-1}\right|+\left|I_n^{-1}-4\tau\sigma^3n^{-1/2}\right| \tag{A.3}$$

and by Lemma B.2 for $n \ge 2$

$$\left|C_n^{-1}-I_n^{-1}\right| \le 16\log n\ \sigma^{-4}I_n^{-1}C_n^{-1}. \tag{A.4}$$

By the Cauchy-Schwarz inequality we have for all $k < m$

$$C_n^{-1} \le A_n(k,m)^{-1} \le \sum_{i=k+1}^{m} t_i^2\frac{\left(\sigma^2\lambda_i+\tau^2\right)^2}{\lambda_i^2}, \quad \text{whenever} \quad \sum_{i=k+1}^{m} t_i = 1. \tag{A.5}$$

Hence with Lemma B.1 it follows for k, m, $k \ll n^{1/2} \ll m$, n sufficiently large

$$\begin{aligned}
A_n(k,m)^{-1} &\le \sum_{i=\left[n^{1/2}\right]+1}^{2\left[n^{1/2}\right]}\left[n^{1/2}\right]^{-2}\left(\sigma^2+\tau^2\lambda_i^{-1}\right)^2 \\
&\le 2\left[n^{1/2}\right]^{-1}\left(\sigma^4+\tau^4\lambda_{2\left[n^{1/2}\right]}^{-2}\right) \le 4n^{-1/2}\left(\sigma^4+16\pi^4\tau^4\right)
\end{aligned} \tag{A.6}$$

and $C_n^{-1} \leq 4n^{-1/2}\left(\sigma^4 + 16\pi^4\tau^4\right)$. We now estimate $(\sigma\tau)^{-5}\left|C_n^{-1} - I_n^{-1}\right|$ using (A.4) and $(a+b)^r \leq 2^r\left(a^r + b^r\right)$ for $a, b, r \geq 0$, as

$$\sup_{\sigma,\tau>\epsilon} \frac{16\log n}{\sigma^9\tau^5} I_n^{-1}C_n^{-1} \leq \sup_{\sigma,\tau>\epsilon} \frac{2^{7/2}64\log n\left(\sigma^7 + 2^7 n^{7/2}\tau^7\right)n^{-1/2}\left(\sigma^4 + 16\pi^4\tau^4\right)}{\sigma^6\tau^5\left(32n^3\tau^4\sigma^2 + 10n^2\tau^2\sigma^4 + 32n^4\tau^6 + n\sigma^6\right)}$$

and some elementary calculations finally yield

$$\sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^5\tau^5}\left|C_n^{-1} - I_n^{-1}\right| = O\left(n^{-1}\log n\right).$$

Note in order to bound the second term in (A.3)

$$\begin{aligned}
\left|I_n^{-1} - 4\tau\sigma^3 n^{-1/2}\right| &\leq \left|\frac{\left(\sigma^2 + 4n\tau^2\right)^{7/2}\sigma^3}{32n^3\tau^4\sigma^2 + 10n^2\tau^2\sigma^4 + 32n^4\tau^6 + n\sigma^6} - 4\tau\sigma^3 n^{-1/2}\right| \\
&\leq \left|\frac{\left(\left(\sigma^2 + 4n\tau^2\right)^{7/2} - \left(4n\tau^2\right)^{7/2}\right)\sigma^3}{32n^3\tau^4\sigma^2 + 10n^2\tau^2\sigma^4 + 32n^4\tau^6 + n\sigma^6}\right| \\
&\quad + \left|\frac{\left(4n\tau^2\right)^{7/2}\sigma^3}{32n^3\tau^4\sigma^2 + 10n^2\tau^2\sigma^4 + 32n^4\tau^6 + n\sigma^6} - 4\tau\sigma^3 n^{-1/2}\right|.
\end{aligned}$$

Using (A.2) yields

$$\begin{aligned}
&\sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^5\tau^5}\left|\frac{\left(\left(\sigma^2 + 4n\tau^2\right)^{7/2} - \left(4n\tau^2\right)^{7/2}\right)\sigma^3}{32n^3\tau^4\sigma^2 + 10n^2\tau^2\sigma^4 + 32n^4\tau^6 + n\sigma^6}\right| \\
&\quad \leq 2^{5/2}7\sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^5\tau^5}\left|\frac{\sigma^{10} + 2^5 n^{5/2}\tau^5\sigma^5}{32n^4\tau^6 + n\sigma^6}\right| = O\left(n^{-1}\right).
\end{aligned}$$

Finally,

$$\begin{aligned}
&\sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^5\tau^5}\left|\frac{\left(4n\tau^2\right)^{7/2}\sigma^3}{32n^3\tau^4\sigma^2 + 10n^2\tau^2\sigma^4 + 32n^4\tau^6 + n\sigma^6} - 4\tau\sigma^3 n^{-1/2}\right| \\
&= \sup_{\sigma,\tau>\epsilon} \frac{4n^{-1/2}}{\sigma^2\tau^4}\left|\frac{32n^3\tau^4\sigma^2 + 10n^2\tau^2\sigma^4 + n\sigma^6}{32n^3\tau^4\sigma^2 + 10n^2\tau^2\sigma^4 + 32n^4\tau^6 + n\sigma^6}\right| = O\left(n^{-1}\right).
\end{aligned}$$

(ii) Note that since $C_n^{-1} \leq A_n^{-1}$ and due to *(i)*

$$\begin{aligned}
&\sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^8\tau^8}\left|A_n^{-1} - 4\tau\sigma^3 n^{-1/2}\right| \\
&\quad \leq \sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^8\tau^8}\left|A_n^{-1} - C_n^{-1}\right| + \sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^8\tau^8}\left|C_n^{-1} - 4\tau\sigma^3 n^{-1/2}\right| \\
&\quad \leq \sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^8\tau^8}\left(C_n - A_n\right)A_n^{-2} + o\left(n^{-1/2}\right).
\end{aligned}$$

By (A.6) it holds further for sufficiently large n

$$C_n - A_n = \sum_{i=1}^{k} \left(\sigma^2 + \tau^2 \lambda_i^{-1}\right)^{-2} + \sum_{i=m+1}^{n} \left(\sigma^2 + \tau^2 \lambda_i^{-1}\right)^{-2} \leq \sigma^{-4} k + \tau^{-4} \sum_{i=m+1}^{n} \lambda_i^2.$$

This finally yields applying Lemma B.1 again

$$\begin{aligned} &\sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^8 \tau^8} \left| A_n^{-1} - C_n^{-1} \right| \\ &\quad \leq \sup_{\sigma,\tau>\epsilon} \left(\sigma^{-4} k + \tau^{-4} \sum_{i=m+1}^{n} \lambda_i^2 \right) 16 n^{-1} \left(\tau^{-4} + 16\pi^4 \sigma^{-4}\right)^2 = o\left(n^{-1/2}\right). \end{aligned}$$

The second statement follows directly from (A.6). □

Lemma A.2. *Let* $k = \left[n^{1/2-b}\right]$ *and* $m = \left[n^{1/2+b}\right]$, $0 < b < 1/2$. *Then, for any* $\epsilon > 0$

$$\sup_{\sigma,\tau>\epsilon} (\sigma\tau)^{-8} \left| \mathrm{E}\left(\hat{A}_n^{-1}\right) - A_n^{-1} \right| = o\left((nk)^{-1/2}\right).$$

Proof. Arguing as in (A.5) yields

$$\hat{A}_n^{-1} \leq \sum_{i=k+1}^{m} t_i^2 \frac{\left(\bar{\sigma}^2 \lambda_i + \hat{\tau}^2\right)^2}{\lambda_i^2}, \quad \text{whenever} \quad \sum_{i=k+1}^{m} t_i = 1$$

and with the choice $t_i = A_n^{-1} \lambda_i^2 / \left(\sigma^2 \lambda_i + \tau^2\right)^2$, $i = k+1, \ldots, m$ we have

$$\hat{A}_n^{-1} \leq A_n^{-2} \sum_{i=k+1}^{m} \frac{\lambda_i^2}{\left(\sigma^2 \lambda_i + \tau^2\right)^4} \left(\bar{\sigma}^2 \lambda_i + \hat{\tau}^2\right)^2.$$

Hence with similar arguments as in (2.18)

$$\begin{aligned} \mathrm{E}\left(\hat{A}_n^{-1}\right) &\leq A_n^{-1} \left(1 + k^{-1/2}\right) \\ &\quad + 2\left(1 + k^{1/2}\right) A_n^{-2} \sum_{i=k+1}^{m} \frac{\lambda_i^2}{\left(\sigma^2 \lambda_i + \tau^2\right)^4} \left(\mathrm{MSE}\left(\bar{\sigma}^2\right) \lambda_i^2 + \mathrm{MSE}\left(\hat{\tau}^2\right)\right) \\ &\leq A_n^{-1} \left[1 + k^{-1/2} + 2\left(1 + k^{1/2}\right) \left(\frac{1}{\sigma^4} \mathrm{MSE}\left(\bar{\sigma}^2\right) + \frac{1}{\tau^4} \mathrm{MSE}\left(\hat{\tau}^2\right)\right)\right]. \end{aligned}$$

It follows from (2.12) and (2.13) that for sufficient large n,

$$\mathrm{Bias}^2\left(\bar{\sigma}^2\right) \leq \mathrm{Bias}^2\left(\hat{\tau}^2\right), \quad \mathrm{Var}\left(\bar{\sigma}^2\right) \leq \mathrm{Var}\left(\hat{\tau}^2\right) + \frac{2}{k}\left(\sigma^2 + \tau^2\right)^2$$

and hence by (2.6)

$$\sup_{\sigma,\tau>\epsilon} \frac{1}{\sigma^4\tau^4} \operatorname{MSE}\left(\bar{\sigma}^2\right) = O\left(k^{-1}\right). \tag{A.7}$$

This yields for $\sigma, \tau > \epsilon$

$$\begin{aligned}
&\frac{1}{\sigma^8\tau^8} \left| \operatorname{E}\left(\hat{A}_n^{-1}\right) - A_n^{-1} \right| \\
&\quad \leq \left(\frac{1}{\sigma^4\tau^4} A_n^{-1}\right) \left[k^{-1/2}\epsilon^{-8} + 2\left(1 + k^{1/2}\right)\left(\frac{1}{\sigma^8\tau^4} \operatorname{MSE}\left(\bar{\sigma}^2\right) + \frac{1}{\sigma^4\tau^8} \operatorname{MSE}\left(\hat{\tau}^2\right)\right)\right]
\end{aligned}$$

and thus $\sup_{\sigma,\tau>\epsilon} (\sigma\tau)^{-8} \left| \operatorname{E}\left(\hat{A}_n^{-1}\right) - A_n^{-1} \right| = O\left(n^{-1/2}k^{-1/2}\right).$ □

Lemma A.3. *Let $k = \left[n^{1/2-b}\right]$ and $m = \left[n^{1/2+b}\right]$, $0 < b < 1/18$ and define*

$$\gamma_n := \hat{A}_n^{-2} \sum_{i=k+1}^{m} \frac{\lambda_i^2}{(\bar{\sigma}^2\lambda_i + \hat{\tau}^2)^4} \left[(\sigma^2 - \bar{\sigma}^2)^2\lambda_i^2 + (\tau^2 - \hat{\tau}^2)^2\right].$$

Then, for any $\epsilon > 0$

$$\sup_{\sigma,\tau>\epsilon} (\sigma\tau)^{-8} \operatorname{E}(\gamma_n) = O(n^{9b-1}).$$

Proof. We argue with similar techniques as in the proof of Lemma A.2. Note that

$$\begin{aligned}
&\hat{A}_n^{-2} \sum_{i=k+1}^{m} \frac{\lambda_i^2}{(\bar{\sigma}^2\lambda_i + \hat{\tau}^2)^4} \\
&\quad \leq \hat{A}_n^{-1} A_n^{-2} \sum_{j=k+1}^{m} \frac{\lambda_j^2}{(\sigma^2\lambda_j + \tau^2)^4} \left(\bar{\sigma}^2\lambda_j + \hat{\tau}^2\right)^2 \sum_{i=k+1}^{m} \frac{\lambda_i^2}{(\bar{\sigma}^2\lambda_i + \hat{\tau}^2)^4} \\
&\quad \leq A_n^{-1} \max_{j=k+1,\ldots,m} \frac{\left(\bar{\sigma}^2\lambda_j + \hat{\tau}^2\right)^2}{\left(\sigma^2\lambda_j + \tau^2\right)^2} \max_{i=k+1,\ldots,m} \frac{1}{\left(\bar{\sigma}^2\lambda_i + \hat{\tau}^2\right)^2} \\
&\quad \leq A_n^{-1} \frac{\lambda_{k+1}^2}{\lambda_m^2} \frac{1}{\left(\sigma^2\lambda_m + \tau^2\right)^2} \leq A_n^{-1} \frac{\lambda_{k+1}^2}{\lambda_m^2} \frac{1}{\tau^4}
\end{aligned}$$

and in the same way

$$\hat{A}_n^{-2} \sum_{i=k+1}^{m} \frac{\lambda_i^4}{(\bar{\sigma}^2\lambda_i + \hat{\tau}^2)^4} \leq \frac{1}{\sigma^4} \frac{\lambda_{k+1}^2}{\lambda_m^2} A_n^{-1}.$$

This yields with Lemma A.1, (2.6) and (A.7)

$$\sup_{\sigma,\tau>\epsilon}\frac{1}{\sigma^8\tau^8}\,\mathrm{E}\left(\hat{A}_n^{-2}\sum_{i=k+1}^{m}\frac{\lambda_i^2}{(\bar{\sigma}^2\lambda_i+\hat{\tau}^2)^4}\left[(\sigma^2-\bar{\sigma}^2)^2\lambda_i^2+(\tau^2-\hat{\tau}^2)^2\right]\right)$$
$$\leq \sup_{\sigma,\tau>\epsilon}\frac{1}{\sigma^8\tau^8}\left(A_n^{-1}\frac{\lambda_{k+1}^2}{\lambda_m^2}\left(\frac{1}{\sigma^4}\,\mathrm{MSE}\left(\bar{\sigma}^2\right)+\frac{1}{\tau^4}\,\mathrm{MSE}\left(\hat{\tau}^2\right)\right)\right)=O\left(n^{-1/2}\frac{m^4}{k^5}\right).$$

□

Appendix B. Further Technicalities

Lemma B.1. *Let λ_i as defined in (2.2). Then, it holds for all $n\geq 1$ and $i=1,\dots,n$*

$$\pi^{-2}\frac{n}{i^2}\leq\lambda_i\leq 4\frac{n}{i^2}.$$

Proof. It holds $x\pi/2\leq\sin(x\pi)\leq x\pi$ whenever $x\in[0,1/2]$. Set $x_i:=(2i-1)/(4n+2)$. Hence

$$\frac{i^2}{4n}\leq\frac{ni^2\pi^2}{(4n+2)^2}\leq nx_i^2\pi^2\leq\frac{1}{\lambda_i}\leq 4nx_i^2\pi^2\leq\frac{i^2\pi^2}{n}.$$

□

Lemma B.2. *Let $g(x):=1/\left(\sigma^2+4n\tau^2\sin^2(x\pi)\right)^2$. Define $x_i:=(2i-1)/(4n+2)$ and let $\xi_i\in[(i-1)/(2n),i/(2n)]$. Then, it holds for $n\geq 2$*

$$\sum_{i=1}^{n}\left|g(x_i)-g(\xi_i)\right|\leq\frac{16}{\sigma^4}\log n.$$

Proof. Obviously $|g(x_1)-g(\xi_1)|\leq|g(x_1)|+|g(\xi_1)|\leq 2/\sigma^4$. Because $\xi_i\in[(i-1)/(2n),i/(2n)]$ for $i=1,\dots,n$, we have by Taylor expansion for a suitable $\eta_i\in[(i-1)/(2n),i/(2n)]$,

$$\left|g(x_i)-g(\xi_i)\right|\leq\left|g'(\eta_i)\right|\left(\frac{i}{2n}-\frac{i-1}{2n}\right)=4\tau^2\pi\sin(2\eta_i\pi)\,g(\eta_i)^{3/2}.$$

If $x\in[0,1/2]$ then $x\pi/2\leq\sin(x\pi)$. Hence for sufficiently large n

$$\sum_{i=2}^{n}\left|g(x_i)-g(\xi_i)\right|\leq\sum_{i=2}^{n}\frac{4\tau^2\pi\sin(2\eta_i\pi)}{3\sigma^4 4n\tau^2\sin^2(\eta_i\pi)}$$
$$\leq\sum_{i=2}^{n}\frac{8}{3\sigma^4 n\eta_i}\leq\frac{16}{3\sigma^4}\sum_{i=1}^{n}\frac{1}{i}\leq\frac{16}{3\sigma^4}(1+\log n).$$

□

Appendix C. A Central Limit Theorem

Theorem C.1. *Let $\{Z_{mk} : 1 \leq k \leq m\}$ be a triangular array of i.i.d. random variables with mean* 0 *and variance σ^2 and let c_{mk} be some regression coefficients which satisfy the Noether condition*

(i)

$$\max_{k=1,\ldots,m} |c_{mk}| \to 0.$$

(ii)

$$\sum_{k=1}^{m} c_{mk}^2 \ \to \ C \ , \tag{C.1}$$

where C is a non-zero constant.

Then, it holds that

$$S_m = \sum_{k=1}^{m} c_{mk} Z_{mk} \xrightarrow{\mathcal{D}} N\left(0, C\sigma^2\right).$$

The Noether condition implies Lindeberg's condition and hence the Theorem follows by applying the Lindeberg CLT (Theorem 11.1.1 in Athreya and Lahiri (2006)).

References

Athreya, K. B. and Lahiri, S. N. (2006). *Measure Theory and Probability Theory.* Springer, New York.

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