2. a) Let $Y_i = e^{aX_i}$ for some parameter $a > 0$ to be fixed later, and let $Y = \prod_{i=1}^n Y_i = e^{aX}$. Show that $E(Y) \leq \exp\left(\frac{na^2}{2}\right)$.

By independence, we have $E(Y) = (E(Y_1))^n$. Thus, it suffices to show $E(Y_1) \leq \exp(a^2/2)$. We have

$$E(Y_1) = \frac{1}{2} \cdot (e^a + e^{-a}) = \frac{1}{2} \cdot \sum_{n \in \mathbb{N}} \frac{a^{2n}}{(2n)!} \leq \sum_{n \in \mathbb{N}} \frac{a^{2n}}{(2n)!} \leq \sum_{n \in \mathbb{N}} \frac{a^{2n}}{2^n n!} = \exp(a^2/2).$$

b) This follows from $x \mapsto \exp(ax)$ being strictly monotonically increasing for $a > 0$.

c) This is Markov’s inequality.

d) We choose $a = t/n$. Then $\exp\left(\frac{na^2}{2} - at\right) = \exp\left(\frac{t^2}{2n} - \frac{t^2}{n}\right) = \exp\left(-\frac{t^2}{2n}\right)$.

3. a) This follows from

$$\int_{t}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy = \int_{0}^{\infty} \exp\left(-\frac{(z+t)^2}{2}\right) dz \leq \exp\left(-\frac{t^2}{2}\right) \cdot \int_{0}^{\infty} \exp(-tz) dz = \exp\left(-\frac{t^2}{2}\right) \cdot \frac{1}{t}.$$

b) This follows by scaling and shifting the result from the first part.

4. This can be found in many books on probability theory.

5. a) (This is only a sketch of a proof.) The set $C = \{(x, y) \mid y \geq f(x)\} \subseteq \mathbb{R}^2$ is convex and $f$ is convex. Thus, for every $z \in \mathbb{R}^2 \setminus C$, there exists a separating hyperplane $H_z$ that separates $z$ from $C$. Since $f$ is defined on $\mathbb{R}$, this hyperplane cannot be of the form $H_z = \{(a, y) \mid y \in \mathbb{R}\}$ for some fixed $a \in \mathbb{R}$. Hence, we can write $H_z$ as $H_z = \{(x, ax + b) \mid x \in \mathbb{R}\}$ for some $a, b \in \mathbb{R}$. We put all these $(a, b)$ of these hyperplanes into the set $S$.

b) You can find proofs of Jensen’s inequality in many books on probability theory.