1. After at most $2^k$ iterations (or even 3 iterations), at least one cluster has assumed a third configuration. If the point set is $(\eta, 1)$ coarse, then the potential has decreased by at least $\eta^2$ after these iterations.

The probability that $Y$ is $(\eta, 1)$-coarse is bounded from above by $(7n)^2 \cdot (2n\eta/\sigma)^d$.

For $d \geq 3$, we can bound this probability by $(7n)^2 \cdot (2n\eta/\sigma)^3 = O(n^5\eta^3/\sigma^3)$: either the right factor is at least 1, then the bound is trivial in both cases. Or the right factor is smaller than 1, then decreasing the exponent makes the bound only larger.

If any of at most $2^k$ iterations decreases the potential by less than $\varepsilon$, then $Y$ is not $(\sqrt{\varepsilon}, 1)$-coarse. Thus, the probability there exists a sequence of $2^k$ iterations in which the potential decreases by less than $\varepsilon$ is at most $O(n^5\varepsilon^{1.5}/\sigma^3)$.

The initial potential is at most $O(n^5)$ with a probability of at most $n^{-3kd}$.

We have at least $2^k t$ iterations only if the smallest improvement of any sequence of $2^k$ iterations decreases the potential by at most $n^5/t$, given the bound on the initial potential above.

Then the expected number of sequences of $2^k$ iterations on the point set $Y$ is bounded from above by

$$\sum_{t=1}^{n^{3kd}} O\left(\frac{n^{12.5}}{\sigma^{3.125}}\right) + n^{-3kd} = O\left(\frac{n^{12.5}}{\sigma^3}\right).$$

Multiplying this by $2^k$ (or 3, if we use the better bound on the length of epochs) yields the desired result.

2. We denote by $T(n, \sigma, t)$ the smoothed number of iterations if we have $n$ points, perturb with standard deviation $\sigma$, and the original points come from a hypercube of side length $t$.

By scaling, we observe that $T(n, \sigma, 1) = T(n, t\sigma, t)$ for any $t > 0$.

Furthermore, $T(n, \sigma, t)$ is monotonically increasing in $t$ as increasing $t$ leaves the adversary with more freedom.

Hence, for $\sigma' > \sigma$, we have $T(n, \sigma', 1) = T(n, \sigma, \sigma'/\sigma') \leq T(n, \sigma, 1)$.

3. We refer to the paper “Smoothed Analysis of the $k$-Means Method”, Lemma 4.4. The paper can be found on the website of the course.