1. We prove the following stronger statement: Let \( d_j(v) \) denote the distance in \( G_{f_j} \) from \( s \) to \( v \). Then \( d_0(v) \leq d_1(v) \leq d_2(v) \leq \ldots \) for all nodes \( v \).

We show that \( d_{j-1}(v) \leq d_j(v) \) by double induction, first on \( k \) and then on the depth \( k \) of \( v \) in the shortest path tree of \( G_{f_j} \). For \( k = 0 \), we have \( v = s \), and we have \( d_j(s) = 0 \) for all \( j \). Hence, the statement holds.

Now assume that the statement is true for all nodes of depth at most \( k - 1 \). Let \( v \) be a node of depth \( k \), and let \( u \) be its parent in the shortest path tree of \( G_{f_j} \). Let \( e = (u,v) \).

Note that \( e \) can be a forward or a backward edge.

We have \( d_j(v) = d_j(u) + c_e \). If \( e \) is available in \( G_{f_{j-1}} \), then \( d_{j-1}(v) \leq d_{j-1}(u) + c_e \). If \( e \) is not available in \( G_{f_{j-1}} \), then SSP must have augmented along \( e - 1 \) to obtain \( f_j \). Hence, \( d_{j-1}(u) = d_{j-1}(v) + c_{e-1} = d_{j-1}(v) - c_e \). In both cases, we have \( d_{j-1}(v) \leq d_{j-1}(u) + c_e \).

Now we apply the induction hypothesis for \( u \), which yields \( d_{j-1}(v) \leq d_{j-1}(u) + c_e \leq d_j(u) + c_e = d_j(v) \).

2. If \( G_f \) contains a directed cycle \( C \) with \( c(C) < 0 \), then we can augment along \( C \) and obtain a cheaper flow.

Now assume that \( f \) is not an optimal flow. Let \( f^* \) be an optimal \( b \)-flow. We have to show that \( G_f \) contains a cycle of negative costs.

We consider a circulation \( g \) with \( g(e) = f^*(e) - f(e) \). We call \( g \) a circulation and not a flow since the budget constraints are not satisfied. Instead, we have in-flow equal to out-flow at all nodes. We have \( c(f^*) = c(g) + c(f) \) by construction. Hence, \( c(g) < 0 \) since we have assumed that \( f \) is not optimal.

We decompose \( g \) into simple cycles \( g_1, \ldots, g_k \) for some \( k \). This means that \( g_i \) is a circulation along one simple cycle. By construction, \( c(g) = \sum_{i=1}^{k} c(g_i) \). Thus, there exists an \( i \) with \( c(g_i) < 0 \). This circulation corresponds to a cycle that is present in \( G_f \) and has negative costs.

3. For all \( u \in V \), let \( \pi(u) = -d(u) \), where \( d(u) \) is the length of a shortest \( s-u \) path in \( G_f \).

Consider any edge \( e = (u,v) \). By the triangle inequality, we have \( d(v) \leq d(u) + c(e) \). This implies \( c'(e) \geq 0 \) by the choice of \( \pi \).