Complete at least two of the following three exercises. If you complete all three, the additional points will be added as extra credit to your grade. Importantly, when solving an exercise you may use any of the statements claimed in the previous exercises (and previous homeworks!)

1. (25 points)

**Farkas Lemma:** Assume the following version of the separator theorem. For $H \subseteq \mathbb{R}^n$ an affine subspace and $\mathbb{R}^n_{\geq 0} = \{ x \in \mathbb{R}^n : x \geq 0 \}$ the non-negative orthant, if $H \cap \mathbb{R}^n_{\geq 0} = \emptyset$ then there exists $w \in \mathbb{R}^n$ such that $\max_{x \in H} w \cdot x < \min_{x \in \mathbb{R}^n_{\geq 0}} w \cdot x$.

Complete the following exercises.

a) Let $H$ and $w$ be above, and write $H = W + t$ where $W$ is a linear subspace and $t$ is a shift. Show that $\min_{x \in H} w \cdot x = \max_{x \in H} w \cdot x = w \cdot t$ and $\min_{x \in \mathbb{R}^n_{\geq 0}} w \cdot x = 0$. In particular, show that $w \in W^\perp = \{ y \in \mathbb{R}^n : y \cdot z = 0 \ \forall z \in W \}$ and $w \geq 0$.

b) Let $P = \{ x \in \mathbb{R}^n : Ax \leq b \}$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $P^* = \{ y \in \mathbb{R}^m : A^T y = 0, y \geq 0, b \cdot y = -1 \}$. Use the above separator theorem to show that $P = \emptyset$ iff $P^* \neq \emptyset$, i.e. a point in $P^*$ is the only certificate of infeasibility for $P$. (Hint: associate $P$ with the intersection of the non-negative orthant with an affine space.)

c) Let $P = \{ x \in \mathbb{R}^n : Ax < b \}$, $A, b$ as above, and let

$$P^* = \{ y \in \mathbb{R}^m : y \geq 0, \sum_{i=1}^m y_i = 1, A^T y = 0, b \cdot y \leq 0 \}.$$ 

Extend part (b) to show that $P = \emptyset$ iff $P^* \neq \emptyset$. (Hint: find a polyhedron $P' = \{ y : Cy \leq d \}$ such that $P \neq \emptyset$ iff $P' \neq \emptyset$).

2. (25 points)

**Borgwardt’s Algorithm:** In this exercise, the goal is to flesh out the details of Borgwardt’s algorithm.

We begin with some notation. Let $P = \{ x \in \mathbb{R}^n : Ax \leq 1 \}$, $A \in \mathbb{R}^{m \times n}$ be a polytope (i.e. $P$ is bounded). Let $a_1, \ldots, a_m \in \mathbb{R}^n$ denote the rows of $A$. For $i \in [n]$ and a vector $x \in \mathbb{R}^n$, define $x^i \in \mathbb{R}^n$ by $x^i_j = x_j$ if $j \in [i]$ and $x^i_j = 0$ if $j \in [n] \setminus [i]$. For each $i \in [n]$, let $P^i = \{ x \in \mathbb{R}^n : Ax \leq 1, x_j = 0 \ \forall j \in [n] \setminus [i] \}$. Note that for $x \in P^i$, we have $x^i = x$ and $a_j \cdot x = a_j^i \cdot x^i$, $j \in [m]$, by construction. For $x \in P$, define $S(x) = \{ j \in [m] : a_j \cdot x = 1 \}$ to be the set of tight constraints at $x$. For a real valued function $f : D \rightarrow \mathbb{R}$, we write $\text{argmax}_{x \in D} f(x)$ to denote the maximizer $x$ of $f$ in $D$ when a maximizer exists and is unique.

Our goal is to solve the optimization problem

$$\max_{x \in P} c \cdot x,$$
where \( c \in \mathbb{R}^n \) and \( c_1 \neq 0 \). The Borgwardt algorithm achieves this by following a sequence of \( n - 1 \) shadow simplex paths, where the corresponding projection planes are fixed and do not depend on \( P \). For this algorithm to work, we will need some non-degeneracy assumptions which we detail a bit later.

The algorithm is described abstractly as follows:

a) Compute \( x_i^* = \text{argmax}_{x \in P_i} c^i \cdot x \).

b) For each \( i \in \{2, \ldots, n\} \), do as follows:
   i. Letting \( F_i \) denote the minimum face of \( P^i \) containing \( x_{i-1}^* \) (generically an edge), compute \( v_i = \text{argmax}_{x \in F_i} c^i \cdot x \).
   ii. Compute \( a_i \in \mathbb{R} \) such that \( (c^{i-1} + a_i e_i) \cdot x_{i-1}^* = \max_{x \in P_i} (c^{i-1} + a_i e_i) \cdot x \).
   iii. Follow the shadow simplex path induced by the starting objective \( c^{i-1} + a_i e_i \) and end objective \( c^i \) with starting vertex \( v_i \), to get \( x_i^* = \text{argmax}_{x \in P_i} c^i \cdot x \) (final vertex of the path).

c) Return \( x_n^* \).

For the non-degeneracy assumptions, we assume that:

- For each \( i \in \{n\} \), the maximizer \( x_i^* \) of the program \( \max_{x \in P_i} c^i \cdot x \) is unique.
- For each \( i \in \{n\} \), each vertex \( v \) of \( P^i \) satisfies \( |S(v)| = i \) (exactly dimension many tight constraints).
- For each \( i \in \{2, \ldots, n\} \), the shadow simplex path on \( P^i \) induced by the objectives \( c^{i-1} + a_i e_i \) and \( c^i \) is non-degenerate. In particular, there is a unique sequence of vertices induced by each shadow simplex path.

Complete the following exercises:

a) Give an explicit formula for \( x_i^* \).

b) Step i. Show that the minimum face \( F_i \) of \( P^i \) containing \( x_{i-1}^* \) has dimension at most 1, i.e. either \( x_{i-1}^* \) is a vertex of \( P^i \) or \( x_{i-1}^* \) is in the relative interior of an edge of \( P^i \). Next, give a simple algorithm taking \( i, x_{i-1}^* \) and \( A \) as input, which computes \( v_i \) in polynomial time. (Hint: look at the tight constraints \( S(x_{i-1}^*) \))

c) Step ii. Show that the value \( a_i \) exists and give a simple algorithm to compute it. (Hint: use the tight constraints \( S(x_{i-1}^*) \) and recall LP duality.) Lastly, show that 

\[
(c^i + a_i e_i) \cdot x_i^* = (c^i + a_i e_i) \cdot v_i.
\]

d) Step iii. Argue that the number of simplex pivots performed in step iii is bounded by \( \sum_{i=2}^n |\{ v \in \pi_i(P^i) : v \text{ a vertex of } \pi_i(P^i) \}| \), where \( \pi_i \) is the orthogonal projection onto the subspace spanned by \( c^{i-1} \) and \( e_i \).

3. (25 points)

**Angle Geometry:** Let \( V = (v_1, \ldots, v_k) \in \mathbb{R}^{n \times k} \), such that \( v_i \cdot v_j < 0 \) for \( i, j \in [k] \), \( i \neq j \), i.e. the angle between each pair of vectors is greater than \( \pi/2 \). The main goal of this exercise is to show that for such a set of vectors, \( k \) can be at most \( n + 1 \).

a) Give a simple example with \( k = n + 1 \).
b) Show that for $w \in \text{kernel}(V)$, either $w = 0$ or $w < 0$ or $w > 0$ (Hint: first prove that either $w \geq 0$ or $w \leq 0$).

c) Using (b), show that $\text{kernel}(V)$ has dimension at most 1. Conclude that $k \leq n + 1$.

d) Using (b) and (c), show that if $k = n + 1$ then $\text{cone}(v_1, \ldots, v_{n+1}) := \{ \sum_{i=1}^{n+1} \lambda_i v_i : \lambda_i \geq 0, i \in [n+1] \}$ is equal to $\mathbb{R}^n$. 