Deterministic Algorithms for Multi-Criteria TSP

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Abstract. We present deterministic approximation algorithms for the multi-criteria traveling salesman problem (TSP). Our algorithms are faster and simpler than the existing randomized algorithms.
First, we devise algorithms for the symmetric and asymmetric multi-criteria Max-TSP that achieve ratios of $1/2k - \varepsilon$ and $1/(4k-2) - \varepsilon$, respectively, where k is the number of objective functions. For two objective functions, we obtain ratios of $3/8 - \varepsilon$ and $1/4 - \varepsilon$ for the symmetric and asymmetric TSP, respectively. Our algorithms are self-contained and do not use existing approximation schemes as black boxes.
Second, we adapt the generic cycle cover algorithm for Min-TSP. It achieves ratios of $3/2 + \varepsilon$, $\frac{1}{2} + \frac{\gamma^3}{1-3\gamma^2} + \varepsilon$, and $\frac{1}{2} + \frac{\gamma^2}{1-\gamma} + \varepsilon$ for multi-criteria Min-ATSP with distances 1 and 2, Min-ATSP with γ-triangle inequality and Min-STSP with γ-triangle inequality, respectively.

1 Multi-Criteria TSP

The traveling salesman problem (TSP) is perhaps the best-studied combinatorial optimization problem. An instance of *Min-TSP* is a complete graph $G = (V, E)$ with edge weights $d : E \to \mathbb{Q}_+$ that satisfy the triangle inequality. The goal is to find a *Hamiltonian cycle* (also called a *tour*) of minimum weight, where the weight of a tour is the sum of its edge weights. (The weight of an arbitrary set of edges is defined analogously.) If G is undirected, we have *Min-STSP* (symmetric TSP). If G is directed, we have *Min-ATSP* (asymmetric TSP). If we restrict the problem to instances that fulfill the γ-triangle inequality for $\gamma \in [1/2, 1)$ (this means $d(u,v) \leq \gamma \cdot (d(u,x) + d(x,v))$ for all distinct $u, v, x \in V$), then we get Min-γ-STSP and Min-γ-ATSP. If we restrict the edge weights to 1 and 2, we get *Min-1/2-STSP* and *Min-1/2-ATSP*. For Max-STSP and Max-ATSP, we have edge weights $w : E \to \mathbb{Q}_+$, and the goal is to find a tour of maximum weight.

All these variants of TSP are NP-hard and APX-hard [5]. Thus, we are in need of approximation algorithms. Min-STSP can be approximated with a ratio of 3/2 [5, Sect. 3.1.3]. Min-ATSP allows for a randomized $O(\log n / \log\log n)$ approximation [4] and for a deterministic $\frac{2}{3}\log_2 n$ approximation [13], where n is the number of vertices. Max-STSP and Max-ATSP can be approximated with ratios of 7/9 [22] and 2/3 [17], respectively. Min-γ-STSP and Min-γ-ATSP can be approximated with constant ratios depending on γ [9–11, 25]. Min-1/2-STSP and Min-1/2-ATSP admit factor 8/7 [6] and 5/4 [7] approximations, respectively.

In many scenarios, however, there is more than one objective function to optimize. In case of the TSP, we might want to minimize travel time, expenses, number of flight changes, etc., while we want to maximize, e.g., our profit along the route. This gives rise to multi-criteria TSP, where Hamiltonian cycles are sought that optimize several objectives simultaneously. In order to transfer the notion of an optimal solutions to multi-criteria optimization problems, *Pareto curves* have been introduced (cf. Ehrgott [12]). A Pareto curve is a set of all optimal trade-offs between the different objective functions.

In the following, k always denotes the number of objective functions. We assume throughout the paper that $k \geq 2$ is an arbitrary constant. Let $[k] = \{1, 2, \ldots, k\}$. The k-criteria variants of the TSP that we consider are denoted by k-Min-STSP, k-Min-ATSP, k-Min-γ-STSP, k-Min-γ-ATSP, k-Min-1/2-STSP, k-Min-1/2-ATSP as well as k-Max-STSP and k-Max-ATSP.

We define the following terms for Min-TSP only. After that, we briefly point out the differences for Max-TSP. For a k-criteria variant of Min-TSP, we have edge weights $d_1, \ldots, d_k : E \to \mathbb{Q}_+$. For convenience, let $d = (d_1, \ldots, d_k)$. Inequalities of vectors are meant component-wise. A tour H *dominates* another tour $\tilde{H}$ if $d(H) \leq d(\tilde{H})$ and at least one of these k inequalities is strict. This means that H is strictly preferable to $\tilde{H}$. A *Pareto curve* is a set of all solutions that are not dominated by another solution. Since Pareto curves for the TSP cannot be computed efficiently, we have to be satisfied with approximate Pareto curves. A set $\mathcal{P}$ of tours is called an α *approximate Pareto curve* for the instance (G, d) if the following holds: For every tour $\tilde{H}$ of G, there exists a tour $H \in \mathcal{P}$ of G with $d(H) \leq \alpha d(\tilde{H})$. We have $\alpha \geq 1$, and a 1 approximate Pareto curve is a Pareto curve. An algorithm is called an α *approximation algorithm* if it computes an α approximate Pareto curve.

Let us point out the differences for Max-TSP. We have edge weights $w = (w_1, \ldots, w_k)$ (the triangle inequality is not required). Now a tour H dominates $\tilde{H}$ if $w(H) \geq w(\tilde{H})$ and at least one inequality is strict. A set $\mathcal{P}$ of tours is an α approximate Pareto curve if, for every tour $\tilde{H}$, we have an $H \in \mathcal{P}$ with $w(H) \geq \alpha w(\tilde{H})$. Note that $\alpha \leq 1$ for maximization problems.

1.1 Previous Work

Table 1 shows the current approximation ratios for the different variants of multi-criteria TSP. Many of these approximation algorithms can be extended to the case where some objectives should be minimized and others should be maximized [19]. Unfortunately, no deterministic algorithms are known except for k-Min-STSP and 2-Max-STSP. The reason for this is that most approximation algorithms for multi-criteria TSP use cycle covers. A *cycle cover* of a graph is a set of vertex-disjoint cycles such that every vertex is part of exactly one cycle. Hamiltonian cycles are special cases of cycle covers that consist of just one cycle. In contrast to Hamiltonian cycles, cycle covers of optimal weight can be computed in polynomial time. Cycle covers are one of the main tools for designing approximation algorithms for the TSP [7, 8, 13, 17, 22]. However, only a *randomized* fully polynomial-time approximation scheme (FPTAS) for multi-criteria cycle covers is known [24]. This randomized FPTAS builds on a reduction to a specific unweighted matching problem [23], which is then solved

variant	*randomized*	*deterministic*	*reference*	*new*
2-Min-STSP		2	[15]	
k-Min-STSP		$2+\varepsilon$	[20]	
k-Min-γ-STSP	$\frac{2\gamma^3+2\gamma^2}{3\gamma^2-2\gamma+1}+\varepsilon$, $\frac{1+\gamma}{1+3\gamma-4\gamma^2}+\varepsilon$	$1+\gamma+\varepsilon$, $\frac{2\gamma^2}{2\gamma^2-2\gamma+1}+\varepsilon$	[20]	$\frac{1}{2}+\frac{\gamma^2}{1-\gamma}+\varepsilon$
2-Min-1/2-STSP	$4/3$	$3/2$	[1, 20]	
k-Min-1/2-STSP	$4/3$	$\frac{2k}{k+1}$	[2, 20]	$3/2+\varepsilon$
k-Min-ATSP	$\log n+\varepsilon$		[18]	
k-Min-γ-ATSP	$\frac{1}{1-\gamma}+\varepsilon$		[18]	$\frac{1}{2}+\frac{\gamma^3}{1-3\gamma^2}+\varepsilon$
k-Min-1/2-ATSP	$3/2$		[20]	$3/2+\varepsilon$
2-Max-STSP	$2/3-\varepsilon$	$7/27$	[18, 22]	$3/8-\varepsilon$
k-Max-STSP	$2/3-\varepsilon$		[18]	$\frac{1}{2k}-\varepsilon$
2-Max-ATSP	$1/2$		[14]	$1/4-\varepsilon$
k-Max-ATSP	$1/2$		[14]	$\frac{1}{4k-2}-\varepsilon$

Table 1. Approximation ratios for multi-criteria TSP. The new deterministic ratio for k-Min-γ-STSP is an improvement for $\gamma \leq 0.58$. The new ratio for k-Min-γ-ATSP is achieved for $\gamma < 1/\sqrt{3}$. The result for k-Min-1/2-STSP is an improvement for $k \geq 3$.

using the RNC algorithm by Mulmuley et al. [21]. Derandomizing this algorithm is assumed to be difficult [3], and these nested reductions make the algorithm quite slow. Hence, it is natural to ask whether there exist deterministic, faster approximation algorithms for multi-criteria TSP.

1.2 New Results

We present deterministic approximation algorithms for several variants of multi-criteria TSP. Our algorithms are considerably simpler and faster than the existing randomized approximation algorithms. Table 1 shows an overview.

First, we devise deterministic and self-contained algorithms for Max-TSP (Sect. 2 and 3). They do not use other algorithms as black boxes except for maximum-weight matching with a single objective function. Furthermore, they do not make any assumption about the representation of the edge weights. The existing algorithms require the (admittedly weak and natural) assumption that the edge weights are encoded in binary. For k-Max-ATSP, we get a ratio of $\frac{1}{4k-2}-\varepsilon$ for any $\varepsilon > 0$ (Sect. 2). For k-Max-STSP, we achieve a ratio of $\frac{1}{2k}-\varepsilon$ (Sect. 3). For the special case of two objective functions, we can improve this to $1/4-\varepsilon$ for 2-Max-ATSP and $3/8-\varepsilon$ for 2-Max-STSP. The latter is an improvement over the existing deterministic 7/27 approximation for 2-Max-STSP [18, 22].

Second, we consider the cycle cover algorithm for Min-TSP (Sect. 4). We use a deterministic matching algorithm of Grandoni et al. [16]. The difficulty is that their algorithm does not produce perfect matchings. For k-Min-γ-ATSP, k-Min-γ-STSP, and k-Min-1/2-ATSP, we nevertheless get ratios of $\frac{1}{2}+\frac{\gamma^3}{1-3\gamma^2}+\varepsilon$, $\frac{1}{2}+\frac{\gamma^2}{1-\gamma}+\varepsilon$, and $3/2+\varepsilon$, respectively. The ratio for k-Min-γ-STSP is an improvement over existing algorithms for $\gamma \leq 0.58$. The result for k-Min-γ-ATSP holds for $\gamma < 1/\sqrt{3}$. The result for k-Min-1/2-ATSP holds of course also for k-Min-1/2-STSP, and it is an improvement for $k \geq 3$.

Due to space limitations, many proofs are omitted from this extended abstract.

2 Max-ATSP

The rough idea behind our algorithm for k-Max-ATSP is as follows: First, we "guess" a few edges that we contract to get a slightly smaller instance. The number of edges that we have to contract depends only on k and ε. Second, we compute k maximum-weight matchings in the smaller instance, each with respect to one of the k objective functions. Third, we compute another matching that uses only edges of the k matchings and that contains as much weight as possible with respect to each objective function. One note is here in order: Usually, cycle covers instead of matchings are used for Max-ATSP. However, although the weight of a cycle cover can be (roughly) twice as large as the weight of a maximum-weight matching, we do not get a better approximation ratio by using cycle covers. The reason is that we lose a factor of roughly 1/2 if we compute a collection of paths from k initial cycle covers compared to k initial matchings.

The following lemma is a key ingredient of our algorithm. It shows how to get a matching from k different matchings such that a significant fraction of the weight with respect to each matching is preserved. This works as long as no single edge contributes too much weight. The lemma immediately gives a polynomial-time algorithm for this task.

Lemma 1. *Let $G = (V, E)$ be a directed graph, and let $w = (w_1, \ldots, w_k)$ be edge weights. Let $M_1, \ldots, M_k \subseteq E$ be matchings. Let $\eta \in (0, 1)$ be arbitrary such that $w_i(e) \leq \frac{\eta}{2k-2} \cdot w_i(M_i)$ for all $e \in M_i$ and all $i \in [k]$. Then there exists a matching $P \subseteq \bigcup_{i=1}^{k} M_i$ such that $w_i(P) \geq \frac{1-\eta}{2k-1} \cdot w_i(M_i)$ for all $i \in [k]$. Such a matching P can be computed in polynomial time.*

Proof. We construct the matching as follows: We add one heaviest edge $e \in M_1$ with respect to w_1 to P and remove e and all edges adjacent to e from $M_2, \ldots, M_k$. Then we put one heaviest remaining edge from M_2 into P and remove it and all adjacent edges. We proceed with $M_3, \ldots, M_k$ and repeat the process until no edges remain.

Let us analyze $w_i(P)$. In each step, at most two edges of any M_i are removed. Thus, we have removed at most $2i - 2$ edges from M_i until we added the first edge from M_i to P. The weight of these edges is at most $(2i-2) \cdot \frac{\eta}{2k-2} w_i(M_i) \leq \eta w_i(M_i)$. Now let e be an edge of M_i that we added to P, and let $e_1, \ldots, e_t$ be the $t \leq 2k - 2$ edges that are removed from M_i in the subsequent rounds of the procedure until again an edge of M_i is added. By construction, we have $w_i(e) \geq w_i(e_j)$ for all $j \in [t]$. Thus, $w_i(e) \geq \frac{1}{2k-1} \cdot (w_i(e) + \sum_{j=1}^{t} w_i(e_j))$. Taking the initial loss of $\eta w_i(M_i)$ into account, we observe that we can put a $\frac{1}{2k-1}$ fraction of $(1 - \eta) w_i(M_i)$ into P for each $i \in [k]$. □

Now we have to make sure that, for a tour $\tilde{H}$, we can find appropriate matchings $M_1, \ldots, M_k$. For a directed complete graph $G = (V, E)$ and a set $K \subseteq E$ that forms a subset of a tour, we obtain G_{-K} by contracting all edges of K. Contracting an edge (u, v) means that we remove all outgoing edges of u and all incoming edges of v, and then identify u and v. Analogously, for a tour $\tilde{H} \supseteq K$, we obtain a tour $\tilde{H}_{-K}$ by contracting the edges in K.

The following lemma says that, for any tour $\tilde{H}$, there is always a small set K of edges such that, if we contract these edges, the resulting tour $\tilde{H}_{-K}$ consists solely of edges that do not contribute too much to the weight of $\tilde{H}_{-K}$ with respect to any objective function. The proof is identical to the proof of the corresponding lemma for the $(1/2 - \varepsilon)$ approximation for k-Max-ATSP [18, 19]. In the algorithm, we will "guess" good sets K, compute Hamiltonian cycles on G_{-K}, and add the edges of K to get a Hamiltonian cycle of G.

Small set means that $|K| \leq f(k, \varepsilon)$ for some function f that does not depend on the number n of vertices. We can choose $f(k, \varepsilon) \in O(k/\log(1/(1-\varepsilon))) = O(k/\log(1+\varepsilon)) = O(k/\varepsilon)$ [18, 19] (we have $\log(1+\varepsilon) = O(1/\varepsilon)$ by Taylor expansion). Moreover, we can choose K such that V_{-K} contains an even number of vertices.

Lemma 2. *Let $G = (V, E)$ be a directed complete graph with edge weights $w = (w_1, \ldots, w_k)$, and let $\varepsilon > 0$. Let $H \subseteq E$ be any tour of G. Then there is a subset $K \subseteq H$ such that $|K| \leq f(k, \varepsilon)$, $|V_{-K}|$ is even, and, for all $i \in [k]$, we have*

1. *$w_i(K) \geq \frac{1}{4} \cdot w_i(H)$ or*
2. *$w_i(e) \leq \varepsilon \cdot w_i(H_{-K})$ for all $e \in H_{-K}$.*

We have to make sure that any edge weighs at most an ε fraction of $w(H)$, provided that $w(e) \leq \varepsilon w(H)$ for all $e \in H$: Let $\beta_i = \max\{w_i(e) \mid e \in H\}$ be the weight of the heaviest edge with respect to w_i. Let $\beta = (\beta_1, \ldots, \beta_k)$. We define new edge weights w^β by setting the weight of edges that are too heavy to 0:

$$w^\beta(e) = \begin{cases} w(e) & \text{if } w(e) \leq \beta \text{ and} \\ 0 & \text{if } w_i(e) > \beta_i \text{ for some } i. \end{cases}$$

Since $w(e) \leq \beta$ for every $e \in H$ by definition, we have $w(H) = w^\beta(H)$. The number of vectors β that result in different weight functions w^β is bounded by n^{2k}: Since the number of edges is less than n^2, there are less than n^2 different edge weights for each objective function. Now we can state and analyze our approximation algorithm for k-Max-ATSP (Algorithm 1).

Theorem 3. *For every $\varepsilon > 0$ and $k \geq 2$, Algorithm 1 is a deterministic approximation algorithm for k-Max-ATSP that achieves an approximation ratio of $\frac{1}{4k-2} - \varepsilon$. Its running-time is $n^{O(k/\varepsilon)}$.*

Proof. We have to show that, for every tour $\tilde{H}$, there exists a tour $H \in \mathcal{P}_{\text{TSP}}$ with $w(H) \geq (\frac{1}{4k-2} - \varepsilon) \cdot w(\tilde{H})$. By Lemma 2, there exists a subset $K \subseteq \tilde{H}$ of edges and an $I \subseteq [k]$ such that $|K| \leq f(k, \varepsilon)$, $|V_{-K}|$ is even, $w_i(K) \geq w_i(\tilde{H})/4$

$\mathcal{P}_{\text{TSP}} \leftarrow$ MaxATSP-Approx(G, w, ε)
input: directed complete graph $G = (V, E)$, $w : E \to \mathbb{Q}_+^k$, $\varepsilon > 0$
output: $\frac{1}{4k-2} - \varepsilon$ approximate Pareto curve $\mathcal{P}_{\text{TSP}}$ for k-Max-ATSP
for all $K \subseteq E$ that form a subset of a tour with $|K| \leq f(k, \varepsilon)$ and $|V_{-K}|$ even **do**
  for all $I \subseteq [k]$ and β **do**
    compute maximum-weight matchings M_i in G_{-K} w.r.t. w_i^β for $i \in \overline{I} = [k] \setminus I$
    compute a matching $P \subseteq \bigcup_{i \in \overline{I}} M_i$ according to Lemma 1
    add edges to $K \cup P$ to obtain a Hamiltonian cycle H; add H to $\mathcal{P}_{\text{TSP}}$

Algorithm 1: Approximation algorithm for k-Max-ATSP.

for all $i \in I$, and $w_i(e) \leq \varepsilon w_i(\tilde{H}_{-K})$ for all $e \in \tilde{H}_{-K}$ and $i \in [k] \setminus I$. Let $i \in [k] \setminus I$, and let M_i be a maximum-weight matching in G_{-K} with respect to w_i^β. Then $w_i^\beta(M_i) \geq w_i^\beta(\tilde{H}_{-K})/2$ and $w_i^\beta(e) \leq 2\varepsilon w_i(\tilde{H}_{-K})$. Using Lemma 1 with $\eta = (2k-2)2\varepsilon$, we can compute a matching $P \subseteq \bigcup_{i \in [k] \setminus I} M_i$ such that $w_i^\beta(P) \geq \frac{1-\eta}{2k-1} \cdot w_i^\beta(M_i) = \frac{1-(2k-2)2\varepsilon}{2k-1} \cdot w_i^\beta(M_i) \geq (\frac{1}{2k-1} - 2\varepsilon) \cdot w_i^\beta(M_i)$. Now $P \cup K$ is a collection of paths in G. What remains to be done is to estimate the weight of $w(P \cup K)$. For every $i \in I$, we have $w_i(P \cup K) \geq w_i(K) \geq w_i(\tilde{H})/4 \geq \left(\frac{1}{4k-2} - \varepsilon\right) \cdot w_i(\tilde{H})$. For every $i \notin I$, we note that $w_i(\tilde{H}) = w_i(K) + w_i(\tilde{H}_{-K})$. This gives us

$$\begin{aligned} w_i(P \cup K) &\geq w_i^\beta(P) + w_i(K) \geq \left(\tfrac{1}{2k-1} - 2\varepsilon\right) \cdot w_i^\beta(M_i) + w_i(K) \\ &\geq \left(\tfrac{1}{4k-2} - \varepsilon\right) \cdot w_i^\beta(\tilde{H}_{-K}) + w_i(K) \geq \left(\tfrac{1}{4k-2} - \varepsilon\right) w_i(\tilde{H}). \end{aligned}$$

The running-time is at most $n^{O(1)+2k+f(k,\varepsilon)} = n^{O(k/\varepsilon)}$. □

If we have only two objective functions, we can improve the approximation ratio to $1/4 - \varepsilon$. The key ingredient for this is the following lemma, which is the improved counterpart of Lemma 1 for $k = 2$. The lemma can be proved using a cake-cutting argument with one player for each of the two objective functions.

Lemma 4. *Let $G = (V, E)$ be a directed graph with edge weights $w = (w_1, w_2)$ and an even number of vertices. Let $M_1, M_2 \subseteq E$ be two perfect matchings, and let $\eta \in (0, 1/4)$. Suppose that $w_i(e) \leq \frac{\eta}{2} \cdot w_i(M_i)$ for all $e \in M_i$ and $i \in \{1, 2\}$. Then there is a matching $P \subseteq M_1 \cup M_2$ with $w_i(P) \geq (\frac{1}{2} - \sqrt{\eta}) w_i(M_i)$ for $i \in \{1, 2\}$. The matching P can be found in polynomial time.*

Proof. Without loss of generality, we assume $M_1 \cap M_2 = \emptyset$. Otherwise, we can simply remove $M_1 \cap M_2$ from both matchings and add it to P. We scale the edge weights such that $w_i(M_i) = 1$ for $i \in \{1, 2\}$. If we ignore the directions of the edges, the graph with edges $M_1 \cup M_2$ is a collection of disjoint cycles. Every cycle has even length and edges from M_1 and M_2 alternate.

Let $c \subseteq M_1 \cup M_2$ be a cycle. We say that c is a light cycle if $w_1(c) \leq \sqrt{\eta}$. Otherwise, i.e., if $w_1(c) > \sqrt{\eta}$, we call c a heavy cycle. Note that $M_1 \cup M_2$ has at most $1/\sqrt{\eta}$ heavy cycles.

We show the lemma by a cake-cutting argument: Player 1 puts cycles (or parts of cycles) into two sets S_1 and S_2, and then Player 2 can choose which set

$\mathcal{P}_{\text{TSP}} \leftarrow$ MAXATSP-APPROX-2(G, w, ε)

input: directed complete graph $G = (V, E)$, $w : E \to \mathbb{Q}_+^2$, $\varepsilon > 0$

output: $\frac{1}{4} - \varepsilon$ approximate Pareto curve $\mathcal{P}_{\text{TSP}}$ for k-Max-ATSP

for all $K \subseteq E$ with $|K| \leq f(2, \varepsilon^2)$ that are a subset of a tour and $|V_{-K}|$ even **do**

  for all $I \subseteq \{1, 2\}$ and β **do**

    compute maximum-weight matchings M_i in G_{-K} w.r.t. w_i^β for $i \in \overline{I}$

    compute a matching $P \subseteq \bigcup_{i \in \overline{I}} M_i$ according to Lemma 4

    add edges to $K \cup P$ to obtain a Hamiltonian cycle H; add H to $\mathcal{P}_{\text{TSP}}$

Algorithm 2: Improved approximation algorithm for 2-Max-ATSP.

to take. Player i wants to maximize w_i. Player 1 puts light cycles as a whole into S_1 or S_2. Heavy cycles are split into two parts as follows: Player 1 decides to remove one edge of M_1 and one edge of M_2 (these edges are lost also for Player 2). In this way, we get two paths (again disregarding the directions of the edges). Player 1 puts one path into S_1 and the other path into S_2. (It can happen that one of the paths is empty: If we have a cycle of length four, the two edges removed are necessarily adjacent. This, however, does not cause any problem. In particular, cycles of length four are always light cycles.) Finally, Player 2 chooses the set S_i that maximizes w_2. Player 1 has to take S_{3-i}. This yields the matching $P = (S_i \cap M_2) \cup (S_{3-i} \cap M_1)$.

Let us estimate the weight that the players are guaranteed to get. Since we have at most $1/\sqrt{\eta}$ heavy cycles, at most $1/\sqrt{\eta}$ edges from M_2 are removed. The total weight of the edges removed is hence at most $\sqrt{\eta}/2$. Thus, $w_2((S_1 \cup S_2) \cap M_2) \geq w_2(M_2) - \sqrt{\eta}/2 = 1 - \sqrt{\eta}/2$. Hence, Player 2 can always get a weight of at least $\frac{1}{2} \cdot (1 - \sqrt{\eta}/2) \geq \frac{1}{2} - \sqrt{\eta}$.

Let us now focus on Player 1. As for Player 2, we have $w_1((S_1 \cup S_2) \cap M_1) \geq 1 - \sqrt{\eta}/2$. For any heavy weight cycle c, Player 1 can choose to remove edges such that the resulting paths differ by at most $\eta/2$ with respect to w_1. Since light cycles are put as a whole in either S_1 or S_2 and have a weight of at most $\sqrt{\eta}$ with respect to w_1, Player 1 can make sure that $w_1(S_1 \cap M_1)$ and $w_1(S_2 \cap M_1)$ differ by at most $\sqrt{\eta}$. Thus, $w_1(S_i \cap M_1) \geq \frac{1}{2} \cdot \left(1 - \frac{\sqrt{\eta}}{2}\right) - \frac{\sqrt{\eta}}{2} \geq \frac{1}{2} - \sqrt{\eta}$ for both $i \in \{1, 2\}$. Thus, for any choice of Player 2, Player 1 still gets enough weight with respect to w_1. The proof immediately gives a polynomial-time algorithm for computing P. □

Theorem 5. *For every $\varepsilon > 0$, Algorithm 2 is a deterministic approximation algorithm for 2-Max-ATSP with an approximation ratio of $1/4 - \varepsilon$. Its running-time is $n^{O(1/\varepsilon^2)}$.*

Proof. We have to prove that, for every tour $\tilde{H}$, there is an $H \in \mathcal{P}_{\text{TSP}}$ with $w(H) \geq (\frac{1}{4} - \varepsilon) \cdot w(\tilde{H})$. According to Lemma 2, there is a subset $K \subseteq \tilde{H}$ and an $I \subseteq \{1, 2\}$ such that $|K| \leq f(2, \varepsilon^2)$, $|V_{-K}|$ is even, $w_i(K) \geq w_i(\tilde{H})/4$ for $i \in I$, and $w_i(e) \leq \varepsilon^2 w_i(H_{-K})$ for all $e \in H_{-K}$ and $i \in \{1, 2\} \setminus I = \overline{I}$. Thus, there exists a β such that, first, $w_i^\beta(\tilde{H}_{-K}) = w_i(\tilde{H}_{-K})$ for all $i \in \overline{I}$ and, second, for each $i \in \overline{I}$, there exists a matching M_i with $w_i^\beta(e) \leq 2\varepsilon^2 w_i^\beta(M_i)$ and $w^\beta(M_i) \geq \frac{1}{2} \cdot w^\beta(\tilde{H}_{-K})$. Using Lemma 4 with $\eta = 4\varepsilon^2$, we can compute a matching $P \subseteq \bigcup_{i \in \overline{I}} M_i$ such that $w_i^\beta(P) \geq (\frac{1}{2} - 2\varepsilon) w_i^\beta(M_i)$ for each $i \in \overline{I}$. Again, $P \cup K$ is a collection of paths. For any $i \in I$, we have $w_i(P \cup K) \geq w_i(K) \geq w_i(\tilde{H})/4$, which is sufficient. For any $i \in \overline{I}$, we have

$$\begin{aligned} w_i(P \cup K) &\geq w_i^\beta(P) + w_i(K) \geq \left(\tfrac{1}{2} - 2\varepsilon\right) \cdot w_i^\beta(M_i) + w_i(K) \\ &\geq \left(\tfrac{1}{4} - \varepsilon\right) \cdot w_i^\beta(\tilde{H}_{-K}) + w_i(K) \geq \left(\tfrac{1}{4} - \varepsilon\right) \cdot w_i(\tilde{H}). \end{aligned}$$

The running-time is bounded by $n^{O(1) + f(2, 2\varepsilon^2)} = n^{O(1/\varepsilon^2)}$. □

3 Max-STSP

One key ingredient for our algorithm for k-Max-STSP is the following lemma, which is the undirected counterpart to Lemma 1. In contrast to k-Max-ATSP, we now start with k cycle covers rather than k matchings.

Lemma 6. *Let $G = (V, E)$ be an undirected graph with edge weights $w = (w_1, \ldots, w_k)$, and let $C_1, \ldots, C_k \subseteq E$ be cycle covers. Assume that, for some $\eta > 0$, we have $w_i(e) \leq \frac{\eta}{2k-1} w_i(C_i)$ for all $e \in C_i$ and all $i \in [k]$. Then there exists a collection $P \subseteq \bigcup_{i=1}^k C_i$ of paths such that $w_i(P) \geq \frac{1-\eta}{2k} w_i(C_i)$ for all i. Such a collection P can be computed in polynomial time.*

As in Sect. 2, we would like to keep a set $K \subseteq E$ of heavy edges. Unfortunately, it is impossible to contract edges in the same way as in directed graphs [18]. As already done for the randomized algorithms, we circumvent this by setting the weight along paths of sufficient length to 0 [18, 19]. To do this formally, we need the following notation: Let $\tilde{H}$ be a Hamiltonian cycle, and let $K \subseteq \tilde{H}$. Let $L = L(K) = \{v \mid \exists e \in K : v \in e\}$ be the set of vertices that are adjacent to edges of K. Let $T = T(K) = \{e \in \tilde{H} \mid e \text{ is adjacent to } K \text{ but not in } K\}$. As for the directed case, let $\beta = (\beta_1, \ldots, \beta_k)$. Now we define

$$w^{-L,\beta}(e) = \begin{cases} w(e) & \text{if } e \cap L = \emptyset \text{ and } w(e) \leq \beta \text{ and} \\ 0 & \text{if } e \cap L \neq \emptyset \text{ or there is an } i \text{ with } w_i(e) > \beta_i. \end{cases}$$

This means that under $w^{-K,\beta}$, all edges of K or adjacent to K have weight 0. Furthermore, all edges that exceed β for some objective are also set to 0.

Now we are prepared to state the undirected counterpart of Lemma 2. As Lemma 2, its proof is identical to the proof of the corresponding lemma for the $(\frac{2}{3} - \varepsilon)$ approximation for k-Max-STSP [18, 19]. We can choose the function g in the lemma such that $g(k, \eta) \in O\left(\frac{k^3}{\eta \cdot (\log(1-\eta))^2}\right) = O(k^3/\eta^3)$. We can easily require that $|V_{-K}|$ is even. The necessary change of the function g is negligible.

Lemma 7. *Let $G = (V, E)$ be an undirected complete graph with edge weights $w = (w_1, \ldots, w_k)$. Let $\eta > 0$. Let $H \subseteq E$ be any Hamiltonian cycle of G. Then*

$P \leftarrow$ MaxSTSP-Approx(G, w, ε)
input: undirected complete graph $G = (V, E)$, $w : E \to \mathbb{Q}_+^k$, $\varepsilon > 0$
output: $\frac{1}{2k} - \varepsilon$ approximate Pareto curve $\mathcal{P}_{\text{TSP}}$ for k-Max-STSP
for all $K \subseteq E$ with $|K| \leq g(k, \varepsilon/2)$ that form a subset of a tour **do**
  for all $I \subseteq [k]$, and β **do**
    compute maximum-weight cycle covers C_i in G w.r.t. $w_i^{-K,\beta}$ for $i \in \overline{I}$
    compute a collection $P \subseteq \bigcup_{i \in [k] \setminus I} C_i$ of paths according to Lemma 6
    remove edges incident to $L(K)$ from P to obtain P'
    add edges to $K \cup P'$ to obtain a Hamiltonian cycle H; add H to $\mathcal{P}_{\text{TSP}}$

Algorithm 3: $\frac{1}{2k} - \varepsilon$ approximation for k-Max-STSP.

$\mathcal{P}_{\text{TSP}} \leftarrow$ MaxSTSP-Approx-2(G, w, ε)
input: undirected complete graph $G = (V, E)$, $w : E \to \mathbb{Q}_+^2$, $\varepsilon > 0$
output: $\frac{3}{8} - \varepsilon$ approximate Pareto curve $\mathcal{P}_{\text{TSP}}$ for k-Max-STSP
for all $K \subseteq E$ with $|K| \leq g(2, \varepsilon/2)$ that form a subset of a tour **do**
  for all $I \subseteq \{1, 2\}$ and β **do**
    compute maximum-weight matchings M_i in G w.r.t. $w_i^{-K,\beta}$ for $i \in \overline{I}$
    compute a collection $P \subseteq \bigcup_{i \in [k] \setminus I} M_i$ of paths according to Lemma 6
    remove edges incident to $L(K)$ from P to obtain P'
    add edges to $K \cup P'$ to obtain a Hamiltonian cycle H; add H to $\mathcal{P}_{\text{TSP}}$

Algorithm 4: Improved approximation for 2-Max-STSP.

there exists a collection $K \subseteq H$ of paths such that $|K| \leq g(k, \eta)$ and $|V_{-K}|$ is even and the following properties hold: Let $L = L(K)$ and $T = T(K)$. For all $i \in [k]$, we have

1. *$w_i(K) \geq \frac{1}{2} \cdot w_i(H)$ or*
2. *$w_i(e) \leq \eta \cdot w_i^{-L}(H)$ for all $e \in H \setminus K$ and $w_i(T) \leq \eta \cdot w_i(H)$.*

Now we are prepared to state and analyze our approximation algorithm for k-Max-STSP (Algorithm 3), and we obtain the following theorem.

Theorem 8. *For every $k \geq 2$ and $\varepsilon > 0$, Algorithm 3 is a deterministic approximation algorithm for k-Max-STSP that achieves an approximation ratio of $\frac{1}{2k} - \varepsilon$ and has a running-time of $n^{O(k^3/\varepsilon^3)}$.*

As for 2-Max-ATSP, we can achieve a better approximation ratio of $3/8 - \varepsilon$ for $k = 2$. This improves over the known deterministic $7/27$ approximation [18, 22].

Lemma 9. *Let $G = (V, E)$ be an undirected graph with edge weights $w = (w_1, w_2)$, and let $M_1, M_2 \subseteq E$ be two matchings. Assume that $w_i(e) \leq \eta w_i(M_i)$ for $i \in \{1, 2\}$ and all edges $e \in M_i$. Then there exists a collection $P \subseteq M_1 \cup M_2$ of paths such that $w_i(P) \geq (\frac{3}{4} - \eta) \cdot w_i(M_i)$ for $i \in \{1, 2\}$. Such a collection P can be found in polynomial time.*

Theorem 10. *For any $\varepsilon > 0$, Algorithm 4 is a deterministic algorithm for 2-Max-STSP with an approximation ratio of $\frac{3}{8} - \varepsilon$. Its running-time is $n^{O(1/\varepsilon^3)}$.*

$\mathcal{P}_{\text{TSP}} \leftarrow \text{GenericATSP}(G, d, \varepsilon)$
input: directed complete graph $G = (V, E)$, $d : E \to \mathbb{Q}_+^k$, $\varepsilon > 0$
output: approximate Pareto curve $\mathcal{P}_{\text{TSP}}$ for k-Min-ATSP
$w_e \leftarrow 1$ for all $e \in E$
compute an $\frac{\varepsilon}{2\beta}$-approximate Pareto curve $\mathcal{C}$ of $\frac{\varepsilon}{2\beta}$-partial cycle covers
for all $C \in \mathcal{C}$ **do**
  break one edge of every cycle of C to obtain a collection P of paths
  join the paths with edges to obtain a Hamiltonian cycle H; put H into $\mathcal{P}_{\text{TSP}}$

Algorithm 5: Generic Approximation for k-Min-TSP.

4 Cycle Cover Algorithm for Min-TSP

Now we consider multi-criteria Min-TSP. In the following, we need the (natural and weak) assumption that the edge weights are encoded in binary. The main idea is to replace the approximation scheme for cycle covers by the bipartite matching algorithm of Grandoni et al. [16]. Their algorithm does the following: Let $G = (V, E)$ be a bipartite graph, let $\varepsilon > 0$ and k be fixed, let $d = (d_1, \ldots, d_k)$ be edge lengths, and let w be edge weights. Let $D_1, \ldots, D_k$ be budgets. Let M_{opt} be a matching that maximizes $w(M_{\text{opt}})$ subject to $d_i(M_{\text{opt}}) \leq D_i$ for all $i \in [k]$. Then their algorithm outputs a matching M with $w(M) \geq (1-\varepsilon)w(M_{\text{opt}})$ and $d_i(M) \leq (1+\varepsilon)D_i$ for all $i \in [k]$. We use this algorithm to compute partial cycle covers. An ε-partial cycle cover of a directed graph is a collection of simple cycles and simple paths that contains at least $(1-\varepsilon) \cdot n$ edges. (A cycle cover in an n vertex graph consists of n edges.) In other words, a partial cycle cover is a subset of a cycle cover. We do this by exploiting that matchings in bipartite graphs stand in one-to-one correspondence to cycle covers in directed graphs. Let $w(e) = 1$ for all edges $e \in E$. Then our goal is simply to maximize the number of edges subject to the budget constraints. We choose all combinations of 0 and $(1+\varepsilon)^\ell$ (with $\ell \in \{-p, \ldots, p\}$ for some polynomial p) for $D_1, \ldots, D_k$ and run the matching algorithm using these $D_1, \ldots, D_k$ for some small enough ε. This yields $(1+\varepsilon)$ approximate Pareto curves for ε-partial cycle covers [24].

Let $\beta_d = \max_{i \in [k], e, e' \in E} \frac{d_i(e)}{d_i(e')}$ be the maximum ratio of heaviest to lightest edge with respect to any objective function. We remark that it is crucial that β_d is bounded by a constant in order to our algorithm work satisfactory. The reason is that we have to be content with ε-partial cycle covers. Due to this, we might incur extra costs proportional to $\varepsilon\beta_d$.

Theorem 11. *Fix any $\varepsilon > 0$, $k \geq 2$, and $\beta \geq 1$. If restricted to instances (G, d) with $\beta_d \leq \beta$, Algorithm 5 is a $\frac{1+\beta}{2} + \varepsilon$ approximation algorithm for k-Min-ATSP.*

For k-Min-1/2-ATSP, we have $\beta_d \leq 2$. For k-Min-γ-ATSP, we have $\beta_d \leq \frac{2\gamma^3}{1-3\gamma^2}$ for $\gamma < 1/\sqrt{3}$ [11], while for k-Min-γ-STSP, we have $\beta_d \leq \frac{2\gamma^2}{1-\gamma}$ for $\gamma < 1$ [10]. Thus, we get the following derandomized algorithms [18, 20].

Corollary 12. *For every $k \geq 2$ and $\varepsilon > 0$, Algorithm 5 is a deterministic approximation algorithm for multi-criteria Min-TSP. It achieves a ratio of* $3/2+$

ε *for k-Min-1/2-ATSP, a ratio of* $\frac{1}{2} + \frac{\gamma^3}{1-3\gamma^2} + \varepsilon$ *for k-Min-*γ*-ATSP for* $\gamma < 1/\sqrt{3}$, *and a ratio of* $\frac{1}{2} + \frac{\gamma^2}{1-\gamma} + \varepsilon$ *for k-Min-*γ*-STSP for* $\gamma < 1$.

5 Open Problems

An obvious question is whether there exists a *deterministic* approximation algorithm for k-Min-ATSP with a non-trivial approximation ratio, which means smaller than $\frac{2}{3} \cdot k \log_2 n$, which is obtained by adding the k weights of each edge to get a single objective function. Furthermore, we would like to know if there are *deterministic* approximation algorithms for k-Max-ATSP and k-Max-STSP that achieve a constant approximation ratio (or at least a ratio of $\omega(1/k)$).

A key step towards improving the deterministic algorithms for multi-criteria Min-TSP would be an approximation scheme for multi-criteria *non-bipartite perfect* matching. Moreover, the algorithms for k-Min-1/2-STSP and k-Min-γ-STSP would yield a better ratio if initialized with undirected cycle covers. However, a derandomization of the randomized FPTAS for general matching [24], which is based on the isolation lemma [21], seems to be difficult [3].

Finally, it is open if there are *deterministic* algorithms for the case where some objectives should be minimized while others should be maximized.

References

1. Eric Angel, Evripidis Bampis, and Laurent Gourvés. Approximating the Pareto curve with local search for the bicriteria TSP(1,2) problem. *Theoretical Computer Science*, 310(1–3):135–146, 2004.
2. Eric Angel, Evripidis Bampis, Laurent Gourvès, and Jérǫme Monnot. (Non-)approximability for the multi-criteria TSP(1,2). In *Proc. 15th Int. Symp. on Fundamentals of Computation Theory (FCT)*, vol. 3623 of *Lecture Notes in Comput. Sci.*, pp. 329–340. Springer, 2005.
3. Vikraman Arvind and Partha Mukhopadhyay. Derandomizing the isolation lemma and lower bounds for circuit size. In *Proc. 11th Int. Workshop on Approximation Algorithms for Combinatorial Optimization Problems (APPROX)*, vol. 5171 of *Lecture Notes in Comput. Sci.*, pp. 276–289. Springer, 2008.
4. Arash Asadpour, Michel X. Goemans, Aleksander Madry, Shayan Oveis Gharan, and Amin Saberi. An $O(\log n/\log\log n)$-approximation algorithm for the asymmetric traveling salesman problem. In *Proc. 21st Ann. ACM-SIAM Symp. on Discrete Algorithms (SODA)*, pp. 379–389. SIAM, 2010.
5. Giorgio Ausiello, Pierluigi Crescenzi, Giorgio Gambosi, Viggo Kann, Alberto Marchetti-Spaccamela, and Marco Protasi. *Complexity and Approximation.* Springer, 1999.
6. Piotr Berman and Marek Karpinski. 8/7-approximation algorithm for $(1,2)$-TSP. In *Proc. 17th Ann. ACM-SIAM Symp. on Discrete Algorithms (SODA)*, pp. 641–648. SIAM, 2006.
7. Markus Bläser. A 3/4-approximation algorithm for maximum ATSP with weights zero and one. In *Proc. 7th Int. Workshop on Approximation Algorithms for Combinatorial Optimization Problems (APPROX)*, vol. 3122 of *Lecture Notes in Comput. Sci.*, pp. 61–71. Springer, 2004.

8. Markus Bläser and Bodo Manthey. Approximating maximum weight cycle covers in directed graphs with weights zero and one. *Algorithmica*, 42(2):121–139, 2005.
9. Markus Bläser, Bodo Manthey, and Jiří Sgall. An improved approximation algorithm for the asymmetric TSP with strengthened triangle inequality. *Journal of Discrete Algorithms*, 4(4):623–632, 2006.
10. Hans-Joachim Böckenhauer, Juraj Hromkovič, Ralf Klasing, Sebastian Seibert, and Walter Unger. Approximation algorithms for the TSP with sharpened triangle inequality. *Information Processing Letters*, 75(3):133–138, 2000.
11. L. Sunil Chandran and L. Shankar Ram. On the relationship between ATSP and the cycle cover problem. *Theoretical Computer Science*, 370(1-3):218–228, 2007.
12. Matthias Ehrgott. *Multicriteria Optimization*. Springer, 2005.
13. Uriel Feige and Mohit Singh. Improved approximation ratios for traveling salesperson tours and paths in directed graphs. In *Proc. 10th Int. Workshop on Approximation Algorithms for Combinatorial Optimization Problems (APPROX)*, vol. 4627 of *Lecture Notes in Comput. Sci.*, pp. 104–118. Springer, 2007.
14. Christian Glaßer, Christian Reitwießner, and Maximilian Witek. Balanced combinations of solutions in multi-objective optimization. arXiv:1007.5475v1 [cs.DS], 2010.
15. Christian Glaßer, Christian Reitwießner, and Maximilian Witek. Improved and derandomized approximations for two-criteria metric traveling salesman. Report 09-076, Rev. 1, Electron. Colloq. on Computational Complexity (ECCC), 2010.
16. Fabrizio Grandoni, R. Ravi, and Mohit Singh. Iterative rounding for multiobjective optimization problems. In *Proc. 17th Ann. European Symp. on Algorithms (ESA)*, vol. 5757 of *Lecture Notes in Comput. Sci.*, pp. 95–106. Springer, 2009.
17. Haim Kaplan, Moshe Lewenstein, Nira Shafrir, and Maxim I. Sviridenko. Approximation algorithms for asymmetric TSP by decomposing directed regular multigraphs. *Journal of the ACM*, 52(4):602–626, 2005.
18. Bodo Manthey. On approximating multi-criteria TSP. In *Proc. 26th Int. Symp. on Theoretical Aspects of Computer Science (STACS)*, pp. 637–648, 2009.
19. Bodo Manthey. Multi-criteria TSP: Min and max combined. In *Proc. 7th Workshop on Approximation and Online Algorithms (WAOA 2009)*, vol. 5893 of *Lecture Notes in Comput. Sci.*, pp. 205–216. Springer, 2010.
20. Bodo Manthey and L. Shankar Ram. Approximation algorithms for multi-criteria traveling salesman problems. *Algorithmica*, 53(1):69–88, 2009.
21. Ketan Mulmuley, Umesh V. Vazirani, and Vijay V. Vazirani. Matching is as easy as matrix inversion. *Combinatorica*, 7(1):105–113, 1987.
22. Katarzyna Paluch, Marcin Mucha, and Aleksander Madry. A 7/9 approximation algorithm for the maximum traveling salesman problem. In *Proc. 12th Int. Workshop on Approximation Algorithms for Combinatorial Optimization Problems (APPROX)*, vol. 5687 of *Lecture Notes in Comput. Sci.*, pp. 298–311. Springer, 2009.
23. Christos H. Papadimitriou and Mihalis Yannakakis. The complexity of restricted spanning tree problems. *Journal of the ACM*, 29(2):285–309, 1982.
24. Christos H. Papadimitriou and Mihalis Yannakakis. On the approximability of trade-offs and optimal access of web sources. In *Proc. 41st Ann. IEEE Symp. on Foundations of Computer Science (FOCS)*, pp. 86–92. IEEE, 2000.
25. Tongquan Zhang, Weidong Li, and Jianping Li. An improved approximation algorithm for the ATSP with parameterized triangle inequality. *Journal of Algorithms*, 64(2-3):74–78, 2009.