Approximation Algorithms for Multi-criteria Traveling Salesman Problems*

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Abstract. In multi-criteria optimization, several objective functions are to be optimized. Since the different objective functions are usually in conflict with each other, one cannot consider only one particular solution as optimal. Instead, the aim is to compute so-called Pareto curves. Since Pareto curves cannot be computed efficiently in general, we have to be content with approximations to them.

We are concerned with approximating Pareto curves of multi-criteria traveling salesman problems (TSP). We provide algorithms for computing approximate Pareto curves for the symmetric TSP with triangle inequality (Δ-STSP), symmetric and asymmetric TSP with strengthened triangle inequality ($\Delta(\gamma)$-STSP and $\Delta(\gamma)$-ATSP), and symmetric and asymmetric TSP with weights one and two (STSP$(1,2)$ and ATSP$(1,2)$).

We design a deterministic polynomial-time algorithm that computes $(1+\gamma+\varepsilon)$-approximate Pareto curves for multi-criteria $\Delta(\gamma)$-STSP for $\gamma \in [\frac{1}{2}, 1]$. We also present two randomized approximation algorithms for multi-criteria $\Delta(\gamma)$-STSP achieving approximation ratios of $\frac{2\gamma^3+\gamma^2+2\gamma-1}{2\gamma^2}+\varepsilon$ and $\frac{1+\gamma}{1+3\gamma-4\gamma^2}+\varepsilon$, respectively. Moreover, we design randomized approximation algorithms for multi-criteria $\Delta(\gamma)$-ATSP (ratio $\frac{1}{2}+\frac{\gamma^3}{1-3\gamma^2}+\varepsilon$ for $\gamma < 1/\sqrt{3}$), STSP$(1,2)$ (ratio $4/3$) and ATSP$(1,2)$ (ratio $3/2$).

The algorithms for $\Delta(\gamma)$-ATSP, STSP$(1,2)$, and ATSP$(1,2)$ as well as one algorithm for $\Delta(\gamma)$-STSP are based on cycle covers. Therefore, we design randomized approximation schemes for multi-criteria cycle cover problems by showing that multi-criteria graph factor problems admit fully polynomial-time randomized approximation schemes.

1 Introduction

In many practical optimization problems, there is not only one single objective function to measure the quality of a solution, but there are several such functions.

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Consider for instance buying a car: We (probably) want to buy a cheap car that is fast and has a good gas mileage. How do we decide which car is the best one for us? Of course, with respect to any single criterion, making the decision is easy. But with multiple criteria involved, there is no natural notion of a best choice. The aim of *multi-criteria optimization* (also called multi-objective optimization or Pareto optimization) is to cope with this problem. To transfer the concept of a best choice to multi-criteria optimization, the notion of *Pareto curves* was introduced (cf. Section 1.1 and Ehrgott [12]). A Pareto curve is a set of solutions that can be considered optimal.

However, for most optimization problems, Pareto curves cannot be computed efficiently. Thus, we have to be content with approximations to them.

The *traveling salesman problem* (TSP) is one of the best-known combinatorial optimization problems [16]. An instance of the TSP is a complete graph with edge weights, and the aim is to find a Hamiltonian cycle (also called a tour) of minimum weight. Since the TSP is NP-hard [14], we cannot hope to always find an optimal tour efficiently. For practical purposes, however, it is often sufficient to obtain a tour that is close to optimal. In such cases, we require *approximation algorithms*, i.e., polynomial-time algorithms that compute such near-optimal tours.

While the approximability of several variants of the single-criterion TSP has been studied extensively in the past decades, not much is known about the approximability of multi-criteria TSP. The classical TSP is about a traveling salesman who has to visit a certain number of cities and return back home in a shortest tour. "Real" saleswomen and salesmen do not face such a simple situation. Instead, while arranging their tours, they have to bear in mind several objectives that are to be optimized. For instance, the distance travelled and the travel time should be minimized while the journey should be as cheap as possible. This gives rise to multi-criteria TSP, for which we design approximation algorithms in this paper.

1.1 Preliminaries

Graphs and Optimization Problems. Let $G = (V, E)$ be a graph (directed or undirected) with edge weights $w : E \to \mathbb{N}$. We define the weight of a subgraph $G' = (V', E')$ of G or a subset E' of the edges of G as the sum of the weights of its edges: $w(G') = w(E') = \sum_{e \in E'} w(e)$. For $k \in \mathbb{N}$, we define $[k] = \{1, 2, \ldots, k\}$.

TSP in general is the following optimization problem: Given a graph with edge weights, find a Hamiltonian cycle, i.e., a cycle that visits every vertex of the graph exactly once, of minimum weight. In this paper, we are concerned with several variants of the TSP, which are defined below. In case of undirected graphs, we speak of the symmetric TSP (STSP), while in case of directed graphs, we refer to the problem as the asymmetric TSP (ATSP).

Definition 1 (TSP). *An instance of* **Δ-STSP** *is an undirected complete graph $G = (V, E)$ with edge weights $w : E \to \mathbb{N}$ that fulfill triangle inequality, i.e., $w(\{u, v\}) \leq w(\{u, x\}) + w(\{x, v\})$ for all distinct vertices $u, v, x \in V$.*

For $\gamma \in [\frac{1}{2}, 1]$, **$\Delta(\gamma)$-STSP** *is the restriction of* Δ-STSP *to instances that satisfy γ-strengthened triangle inequality, i. e., $w(\{u,v\}) \leq \gamma \cdot (w(\{u,x\}) + w(\{x,v\}))$ for all distinct vertices u, v, x.*

STSP(1, 2) *is the special case of* Δ-STSP *where only one and two are allowed as edge weights, i. e., $w : E \to \{1, 2\}$.*

Δ-ATSP, **$\Delta(\gamma)$-ATSP**, *and* **ATSP(1, 2)** *are analogously defined except that the graphs are directed.*

In all variants, Hamiltonian cycles of minimum weight are sought.

For $\gamma = 1$, $\Delta(\gamma)$-STSP becomes Δ-STSP and $\Delta(\gamma)$-ATSP becomes Δ-ATSP. As γ gets smaller, the edge weights become more and more structured. For $\gamma = 1/2$, all edge weights are equal. The γ-strengthened triangle inequality can also be considered as a data-dependent bound [7]: Given an instance of metric TSP, we compute the minimum γ such that the instance fulfills γ-strengthened triangle inequality. If $\gamma < 1$, then we obtain a better performance guarantee for our approximate solution than with triangle inequality alone.

Throughout the paper, a matching always means a perfect matching, i. e., a set of edges such that every vertex is incident to exactly one edge. **Match** denotes the problem of computing a matching of minimum weight. We refer to a matching as the set M of its edges.

The minimum spanning tree problem, denoted by **MST**, is the problem of computing a spanning tree of minimum weight. We refer to a tree as the set T of its edges.

A **cycle cover** of a graph $G = (V, E)$ is a subgraph (V, C) that consists solely of cycles such that every vertex $v \in V$ is part of exactly one cycle. In most cases, we refer to a cycle cover as the set C of its edges. Hamiltonian cycles are cycle covers that consist of only a single cycle. Below we define two optimization problems concerning cycle covers.

Definition 2 (cycle cover problems). *The problem of computing cycle covers of minimum weight in undirected graphs is called* **SCC**. *The directed version of the problem is called* **ACC**.

Multi-criteria Optimization. Let us first formally define what a k-criteria optimization problem is. We assume in the following that the number k of criteria is fixed. The running-times of our algorithms are usually exponential in k. But since k is typically a small number, this does not cause any harm.

Definition 3 (k-criteria optimization problem). *A k-criteria optimization problem Π consists of a set I of instances, a set $\mathrm{sol}(x)$ of feasible solutions for every instance $x \in I$, k objective functions $w_1, \ldots, w_k$, each mapping pairs of $x \in I$ and $y \in \mathrm{sol}(x)$ to $\mathbb{N}$, and k types indicating whether w_i should be minimized or maximized.*

We refer to Ehrgott and Gandibleux [12, 13] for surveys on multi-criteria optimization problems. Throughout this paper, we restrict ourselves to problems where all objective functions should be minimized. The optimization problems defined in Section 1.1 are generalized to their multi-criteria counterparts in the obvious way: We have k objective functions $w_1, \ldots, w_k$, each induced by edge weight functions (to which we also refer as $w_1, \ldots, w_k$) as described. If we have additional restrictions on the edge weights, like triangle inequality, every edge weight function is assumed to fulfill them. In general, the different objective functions are in conflict with each other, i. e., it is impossible to minimize all of them simultaneously. Therefore, the notion of Pareto curves has been introduced.

For the following definitions, let Π be a k-criteria optimization problem as defined above.

Definition 4 (Pareto curve). *A set $\mathcal{P}(x) \subseteq \mathrm{sol}(x)$ is called a Pareto curve of x if for all solutions $z \in \mathrm{sol}(x)$, there exists a solution $y \in \mathcal{P}(x)$ with $w_i(x, y) \leq w_i(x, z)$ for all $i \in [k]$.*

A Pareto curve contains all solutions that might be considered optimal. For completeness, let us mention that Pareto curves are not unique in general: In our definition, it is not forbidden to include dominated solutions in $\mathcal{P}(x)$ (a solution y is dominated if there exists a z with $w_i(x, z) \leq w_i(x, y)$ for all $i \in [k]$ and $w_i(x, z) < w_i(x, y)$ for some $i \in [k]$, i. e., z is strictly better than y). However, if we restrict ourselves to explicitly given sets of solutions, we can easily get rid of such dominated solutions.

For the majority of multi-criteria problems, computing Pareto curves is hard for two reasons: First, many two-criteria problems allow for a reduction from the knapsack problem. Second, Pareto curves are often of exponential size. Therefore, it is natural to consider the idea of an approximation to Pareto curves.

Definition 5 (β-approximate Pareto curve). *Let $\beta \geq 1$. Let $x \in I$ and $\mathcal{P}^{\mathrm{apx}}(x) \subseteq \mathrm{sol}(x)$. The set $\mathcal{P}^{\mathrm{apx}}(x)$ is called a β-approximate Pareto curve for x if, for every $z \in \mathrm{sol}(x)$, there exists a $y \in \mathcal{P}^{\mathrm{apx}}(x)$ with $w_i(x, y) \leq \beta \cdot w_i(x, z)$ for all $i \in [k]$.*

While Pareto curves itself are often of exponential size, it is known that $(1 + \varepsilon)$-approximate Pareto curves of size polynomial in the input size and $1/\varepsilon$ exist if the number k of criteria is fixed [19]. (The size of the approximate Pareto curve is in general exponential in k. The technical restriction is that the objective functions are restricted to assume values of at most $2^{p(|x|)}$ for $x \in I$ and some polynomial p. This is fulfilled for almost all natural optimization problems.)

The above definition leads immediately to the notion of an approximation algorithm for multi-criteria optimization problems.

Definition 6 (approximation algorithm). *Let $\beta \geq 1$. A β-approximation algorithm for Π is an algorithm that, for every input $x \in I$, computes a β-approximate Pareto curve for x in time polynomial in the size $|x|$ of x.*

A randomized β-approximation algorithm for Π is a polynomial-time algorithm that, for every input $x \in I$, computes a set $\mathcal{P}^{\mathrm{apx}}(x) \subseteq \mathrm{sol}(x)$ such that $\mathcal{P}^{\mathrm{apx}}(x)$ is a β-approximate Pareto curve for x with a probability of at least $1/2$.

By executing a randomized approximation algorithm ℓ times, we obtain a β-approximate Pareto curve with a probability of at least $1 - 2^{-\ell}$, i. e., the failure probability tends exponentially to zero: We take the union of all sets of solutions computed in the ℓ iterations and throw away all solutions that are dominated by solutions in the union.

Definition 7 (FPTAS, FPRAS). *An algorithm is a fully polynomial-time approximation scheme (FPTAS) for Π if, on input $x \in I$ and $\varepsilon > 0$, it computes a $(1+\varepsilon)$-approximate Pareto curve in time polynomial in the size of x and $1/\varepsilon$.*

A fully polynomial-time randomized approximation scheme (FPRAS) for Π is a randomized approximation algorithm that, on input $x \in I$ and $\varepsilon > 0$, computes a $(1+\varepsilon)$-approximate Pareto curve in time polynomial in the size of x and $1/\varepsilon$.

Definition 8 (randomized exact algorithm). *A randomized exact algorithm for Π is an algorithm that, on input x, computes a Pareto curve of x in time polynomial in the size of x with a probability of at least* $1/2$.

An optimization problem Π is said to be polynomially bounded if there exists a polynomial p such that the following holds for every objective function w_i of Π: For every instance x and every feasible solution y for x, $w_i(x,y) \leq p(|x|)$ for all $i \in [k]$. An exact algorithm can be obtained from an FPTAS for a polynomially bounded optimization problem.

Lemma 1. *Suppose that Π is polynomially bounded. If there exists an FPTAS for Π, then Π can be solved exactly in polynomial time. If there exists an FPRAS for Π, then there exists a randomized exact algorithm for Π.*

1.2 Previous Results

The approximability of single-criterion TSP has been studied intensively in the past. The currently best approximation ratios for the variants of single-criterion TSP considered in this paper are $3/2$ for Δ-STSP [10], $8/7$ for STSP$(1,2)$ [5], $\min\{\frac{3\gamma^2}{3\gamma^2-2\gamma+1}, \frac{2-\gamma}{3-3\gamma}\}$ for $\Delta(\gamma)$-STSP [8], $0.842 \cdot \log n$ for Δ-ATSP [15], $5/4$ for ATSP$(1,2)$ [6], and $\min\{\frac{1+\gamma}{2-\gamma-\gamma^3}, \frac{\gamma}{1-\gamma}\}$ for $\Delta(\gamma)$-ATSP [7, 9].

While single-criterion optimization problems and their approximation properties have been the subject of a considerable amount of research (cf. Ausiello et al. [3] for a survey), not much is known about the approximability of multi-criteria optimization problems.

Papadimitriou and Yannakakis [19], by applying results of Barahona and Pulleyblank [4], Mulmuley et al. [17], and themselves [18], showed that there exist FPTASs for multi-criteria MST and the multi-criteria shortest path problem and an FPRAS for multi-criteria Match. (More precisely, a fully polynomial RNC scheme.) The results were established by showing that a multi-criteria problem admits an FPTAS if the exact version of the single-criterion problem can be solved in pseudo-polynomial time. The exact version of a single-criterion optimization problem Π is the following decision problem: Given an instance $x \in I$ and a number $W \in \mathbb{N}$, does there exist a solution $y \in \mathrm{sol}(x)$ with $w(x,y) = W$? The exact versions of many single-criterion optimization problems are NP-hard since knapsack can be reduced to them easily. But this does not rule out the possibility of pseudo-polynomial-time algorithms for them.

Multi-criteria TSP has been investigated by Ehrgott [11] and Angel et al. [1, 2]. Ehrgott [11] analyzed a generalization of Christofides' algorithm for Δ-STSP. Instead of considering Pareto curves, he measured the quality of a solution y for an instance x as a norm of the vector $(w_1(x, y), \ldots, w_k(x, y))$. Thus, he encoded the different objective functions into a single one, which reduces the problem to a single-criterion problem. The approximation ratio achieved is between $3/2$ and 2, depending on the norm used to combine the different criteria. However, by encoding all objective functions into a single one, we lose the special properties of multi-criteria optimization problems.

Angel et al. [1] considered two-criteria STSP$(1, 2)$. They presented a $3/2$-approximation algorithm for this problem by using a local search heuristic. Finally, Angel et al. [2] generalized these results to k-criteria STSP$(1, 2)$ by presenting a $2 - \frac{2}{k+1}$-approximation for $k \geq 3$. Although for every fixed k, the approximation ratio is below 2, it converges to 2 as k increases. Thus, the ratio tends to the trivial ratio of 2, which can be achieved by selecting any Hamiltonian cycle. These two are the only papers about the approximability of Pareto curves of multi-criteria TSP we are aware of.

1.3 Our Results

All our results hold for an arbitrary but fixed number of objective functions.

We present a deterministic polynomial-time algorithm that computes $(2+\varepsilon)$-approximate Pareto curves for Δ-STSP (Section 2.1). This is the first efficient algorithm for computing approximate Pareto curves for this problem. In fact, we show the following more general result: If the edge weights satisfy γ-strengthened triangle inequality for $\gamma \in [\frac{1}{2}, 1]$, then the algorithm computes a $(1 + \gamma + \varepsilon)$-approximate Pareto curve for arbitrarily small $\varepsilon > 0$ in polynomial time.

We generalize Christofides' algorithm [10] (cf. Vazirani [22]) to obtain a randomized approximation algorithm for multi-criteria $\Delta(\gamma)$-STSP (Section 2.2). For $\gamma \in [\frac{1}{2}, 1]$, our algorithm achieves an approximation ratio of $\frac{2\gamma^3+\gamma^2+2\gamma-1}{2\gamma^2} + \varepsilon$. For $\gamma = 1$, this yields a ratio of $2 + \varepsilon$.

We consider cycle covers in Section 3. Cycle covers play an important role in the design of approximation algorithms for the TSP. We prove that there exists an FPRAS for computing approximate Pareto curves of multi-criteria cycle covers. Subsequently, we extend this result and show that the multi-criteria variant of the problem of finding graph factors of minimum weight admits an FPRAS, too.

Finally, we analyze a randomized cycle-cover-based algorithm for multi-criteria TSP (Section 4): We start by computing an approximate Pareto curve of cycle covers. Then, for every cycle cover in the set computed, we remove one edge of every cycle and join the paths thus obtained to a Hamiltonian cycle. We analyze the approximation ratio of this algorithm for $\Delta(\gamma)$-STSP (Section 4.1, approximation ratio $\frac{1+\gamma}{1+3\gamma-4\gamma^2} + \varepsilon$ for $\gamma < 1$), $\Delta(\gamma)$-ATSP (Section 4.2, ratio

Algorithm 1 The tree doubling algorithm for multi-criteria Δ-STSP.

Input: undirected complete graph $G = (V, E)$; edge weights $w_i : E \to \mathbb{N}$ ($i \in [k]$); $\varepsilon > 0$

Output: an approximate Pareto curve $\mathcal{P}_{\text{TSP}}^{\text{apx}}$ to the multi-criteria STSP

1: compute a $(1 + \frac{\varepsilon}{2})$-approximate Pareto curve $\mathcal{P}_{\text{MST}}^{\text{apx}}$ for MST on G

2: **for all** trees $T \in \mathcal{P}_{\text{MST}}^{\text{apx}}$ **do**

3: duplicate all edges in T to obtain an Eulerian graph $\tilde{T}$

4: obtain a Hamiltonian cycle S from $\tilde{T}$ by taking shortcuts; put S into $\mathcal{P}_{\text{TSP}}^{\text{apx}}$

5: **end for**

$\frac{1}{2} + \frac{\gamma^3}{1-3\gamma^2} + \varepsilon$ for $\gamma < 1/\sqrt{3}$), STSP$(1,2)$, and ATSP$(1,2)$ (Section 4.3, ratios $4/3$ and $3/2$, respectively).

As far as we know, our algorithms are the first approximation algorithms for Pareto curves for Δ-STSP, $\Delta(\gamma)$-STSP, $\Delta(\gamma)$-ATSP, and ATSP$(1,2)$. Furthermore, we achieve a better approximation ratio for STSP$(1,2)$ than the algorithms by Angel et al. [1, 2] for all k.

2 Metric TSP

In this section, we present two algorithms for Δ-STSP and $\Delta(\gamma)$-STSP. Another approximation algorithm that can be used for approximating $\Delta(\gamma)$-STSP, which is based on computing cycle covers, will be presented in Section 4.

2.1 The Generalized Tree Doubling Algorithm

Consider the following approximation algorithm for single-criterion Δ-STSP, which was first analyzed by Rosenkrantz et al. [20] (cf. Vazirani [22]): First, we compute a minimum spanning tree. Then we duplicate each edge. The result is an Eulerian graph. We obtain a Hamiltonian cycle from this graph by walking along an Eulerian cycle. If we come back to a vertex that we have already visited, we omit it and take a short-cut to the next vertex in the Eulerian cycle. In this way, we obtain an approximation ratio of 2 for single-criterion Δ-STSP. Algorithm 1 is an adaptation of this algorithm to multi-criteria STSP.

Theorem 1. *For all $\gamma \in [\frac{1}{2}, 1]$, Algorithm 1 computes a $(1+\gamma+\varepsilon)$-approximate Pareto curve for multi-criteria $\Delta(\gamma)$-STSP in time polynomial in the input size and $1/\varepsilon$.*

Corollary 1. *Algorithm 1 is a $(2+\varepsilon)$-approximation algorithm for multi-criteria Δ-STSP. Its running-time is polynomial in the input size and $1/\varepsilon$.*

2.2 A Generalization of Christofides' Algorithm

In this section, we generalize Christofides' algorithm to multi-criteria Δ-STSP, which is the best approximation algorithm for single-criterion Δ-STSP known

Algorithm 2 A generalization of Christofides' algorithm for multi-criteria Δ-STSP.

Input: undirected complete graph $G = (V, E)$; edge weights $w_i : E \to \mathbb{N}$ ($i \in [k]$); $\varepsilon > 0$

Output: an approximate Pareto curve $\mathcal{P}_{\mathrm{TSP}}^{\mathrm{apx}}$ to the multi-criteria STSP (with a probability of at least 1/2)

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compute a (1 + ε/2)-approximate Pareto curve P^apx_MST for MST on G
let p be the number of trees in P^apx_MST
for all trees T ∈ P^apx_MST do
    let V_odd ⊆ V be the set of vertices of odd degree in T
    compute P^apx_Match(T) such that P^apx_Match(T) is a (1 + ε/2)-approximate Pareto curve
        for Match on the graph induced by V_odd with a probability of at least 1 − 1/(2p)
    for all matchings M ∈ P^apx_Match(T) do
        obtain a Hamiltonian cycle S from T ∪ M by taking shortcuts; put S into
            P^apx_TSP
    end for
end for
```

so far. This algorithm computes Pareto curves of matchings. In case of single-criterion Δ-STSP, we can always find a matching with a weight of at most half of the weight of the optimal Hamiltonian cycle. This is in contrast to multi-criteria Δ-STSP, where the weights of the matchings can be arbitrarily close to the weight of the optimal Hamiltonian cycle. The reason is that we cannot choose the lighter of two different matchings since multiple objective functions are involved; the term "'lighter"' is not well defined. Therefore, we only get an approximation ratio of roughly two in this case. But for $\Delta(\gamma)$-STSP, we can show a better upper bound. The analysis of the algorithm exploits the following result due to Böckenhauer et al. [8].

Lemma 2 (Böckenhauer et al. [8]). *Let $G = (V, E)$ be an undirected complete graph with an edge weight function w satisfying γ-strengthened triangle inequality for some $\gamma \in [\frac{1}{2}, 1)$.*

Let $w_{\max} = \max_{e \in E}(w(e))$ and $w_{\min} = \min_{e \in E}(w(e))$ be the weights of a heaviest and lightest edge, respectively. Then $\frac{w_{\max}}{w_{\min}} \leq \frac{2\gamma^2}{1-\gamma}$.

Let e and e' be two edges with a common endpoint. Then $\frac{w(e)}{w(e')} \leq \frac{\gamma}{1-\gamma}$.

Theorem 2. *For $\gamma \in [\frac{1}{2}, 1]$, Algorithm 2 is a randomized $\left(\frac{2\gamma^3+\gamma^2+2\gamma-1}{2\gamma^2} + \varepsilon\right)$-approximation algorithm for multi-criteria $\Delta(\gamma)$-STSP. Its running time is polynomial in the input size and $1/\varepsilon$.*

We compare the ratios obtained by the two algorithms of this sections and the cycle cover algorithm of Section 4 in Section 5.1.

3 Matchings and Cycle Covers

A cycle cover of a graph is a spanning subgraph that consists solely of cycles such that every vertex is part of exactly one cycle. Many approximation algorithms for the single-criterion TSP are based on cycle covers. These approximation algorithms usually start by computing an initial cycle cover and then join the cycles to obtain a Hamiltonian cycle. This technique is called *subtour patching* [16].

3.1 Multi-criteria ACC

ACC, the cycle cover problem in directed graphs, is equivalent to finding matchings of minimum weight in bipartite graphs (assignment problem). An FPRAS for multi-criteria Match is also an FPRAS for the multi-criteria matching problem in bipartite graphs. Hence, multi-criteria ACC also admits an FPRAS.

Theorem 3. *There exists an FPRAS for the multi-criteria* ACC*.*

3.2 Multi-criteria SCC and f-factors

To show that multi-criteria SCC admits an FPRAS, we reduce SCC using Tutte's reduction [21] to the matching problem in general graphs. Cycle covers in undirected graphs are also known as two-factors since in a cycle cover, every vertex is incident to exactly two edges. We obtain the following result from the fact that the matching problem admits an FPRAS [19].

Theorem 4. *Multi-criteria* SCC *admits an FPRAS.*

We can generalize the FPRAS for the undirected cycle cover problem to arbitrary f-factors by exploiting Tutte's reduction again. The proof goes along the same lines as the proof of Theorem 4. Let $G = (V, E)$ be an undirected graph and $f : V \to \mathbb{N}$ be any function. A subset $F \subseteq E$ is called an f-factor of G if all $v \in V$ are incident to exactly $f(v)$ edges in F. If $f(v) = 2$ for all $v \in V$, an f-factor is a cycle cover.

Definition 9 (GFP). *The graph factor problem* GFP *is the following minimization problem: An instance is an undirected graph $G = (V, E)$ with a function $f : V \to \mathbb{N}$ and an edge weight function $w : E \to \mathbb{N}$. The aim is to find an f-factor of minimum weight.*

Theorem 5. *Multi-criteria* GFP *admits an FPRAS.*

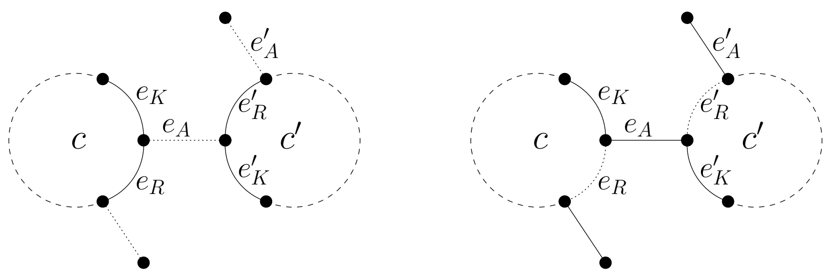
4 Approximations Based on Cycle Covers

The generic outline of a cycle-cover-based algorithm is the following: Start by computing a cycle cover. Then remove one edge of every cycle. Finally, join the paths thus obtained to form a Hamiltonian cycle. Algorithm 3 is our generalization of this algorithm to multi-criteria TSP. It achieves a constant approximation ratio if the quotient of the weight of the heaviest edge and the weight of the lightest edge is bounded. In this section, we present a general analysis of the approximation ratio of this algorithm. We will refine the analysis for multi-criteria $\Delta(\gamma)$-STSP (Section 4.1) to get an improved approximation ratio. Furthermore,

Algorithm 3 An approximation algorithm based on cycle covers for multi-criteria TSP.

Input: complete graph $G = (V, E)$; edge weights w_i $(i \in [k])$; $\varepsilon' > 0$
Output: approximate Pareto curve $\mathcal{P}_{\text{TSP}}^{\text{apx}}$ to multi-criteria TSP (with a probability of at least $1/2$)

compute a $(1+\varepsilon')$-approximate Pareto curve $\mathcal{P}_{\text{CC}}$ to the multi-criteria cycle cover problem
for all cycle covers $C \in \mathcal{P}_{\text{CC}}$ **do**
  for all cycles c of C **do**
    remove one edge of c
  end for
  join the paths to form a Hamiltonian cycle S and add S to $\mathcal{P}_{\text{TSP}}^{\text{apx}}$
end for



(a) Cycle cover, before the patching. (b) Hamiltonian cycle, after the patching.

Fig. 1. Two cycles c and c' before and after joining the cycles to a Hamiltonian cycle. The edges e_R, e_K, and e_A belong to c while e'_R, e'_K, and e'_A belong to c'.

we apply the analysis to get approximation results for multi-criteria $\Delta(\gamma)$-ATSP (Section 4.2) and STSP$(1, 2)$ and ATSP$(1, 2)$ (Section 4.3). We analyze Algorithm 3 in terms of the number αn of edges that have to be removed and the quotient $\beta = w_{\max}/w_{\min}$.

Lemma 3. *Assume that at most αn edges have to be removed from each cycle cover and that $\frac{\max_{e \in E} w_i(e)}{\min_{e \in E} w_i(e)} \leq \beta$ for all $i \in [k]$. Then Algorithm 3 is a randomized $(1 + \alpha(\beta - 1) + \varepsilon)$ approximation for every $\varepsilon > 0$. Its running-time is polynomial in the input size and $1/\varepsilon$.*

4.1 Refined Analysis for $\Delta(\gamma)$-STSP

From the general analysis, we obtain an approximation ratio of $\frac{2}{3} + \frac{1}{3} \cdot \frac{2\gamma^2}{1-\gamma} + \varepsilon$ for $\Delta(\gamma)$-STSP. In this section, we present a refined analysis that yields a better approximation ratio.

Consider any cycle c of a cycle cover of $\mathcal{P}_{\text{CC}}$. There will be an edge e_R of c that will be removed and an edge e_A adjacent to e_R that will be added during the joining process. Finally, there exists an edge e_K of c that is adjacent to both

e_R and e_A (Figure 1 shows an example). Note that while e_R is unique, once the edges have been removed and added, the edge e_A is not since there are two edges that connect c to other cycles. However, once we have fixed e_A for one cycle c, the corresponding e_K is uniquely determined, and the e'_A and e'_K of all other cycles c' are also determined. By Lemma 2, we have $w_i(e_R) \geq \frac{1-\gamma}{\gamma} \cdot w_i(e_A)$ and $w_i(e_K) \geq \frac{1-\gamma}{\gamma} \cdot w_i(e_A)$. Exploiting these inequalities, we obtain the following result.

Theorem 6. *Algorithm 3 is a randomized* $\left(\frac{1+\gamma}{1+3\gamma-4\gamma^2} + \varepsilon\right)$*-approximation algorithm for all* $\varepsilon > 0$*. Its running-time is polynomial in the input size and* $1/\varepsilon$*.*

In Section 5.1, we compare the approximation ratios of the cycle cover algorithm for $\Delta(\gamma)$-STSP to the tree doubling and Christofides' algorithm.

4.2 The Cycle Cover Algorithm for $\Delta(\gamma)$-ATSP

For multi-criteria $\Delta(\gamma)$-ATSP, our algorithm yields a constant factor approximation if $\gamma < 1/\sqrt{3}$ since $w_{\max}/w_{\min}$ is bounded from above by $\frac{2\gamma^3}{1-3\gamma^2}$ for such γ. For larger values of γ, this ratio can be unbounded.

Lemma 4 (Chandran and Ram [9]). *Let* $\gamma \in [1/2, 1)$*. Let* $G = (V, E)$ *be a directed complete graph, and let* $w : E \to \mathbb{N}$ *be an edge weight function satisfying* γ*-triangle inequality. Let* $w_{\min} = \min_{e \in E} w(e)$ *and* $w_{\max} = \max_{e \in E} w(e)$*.*

If $\gamma < 1/\sqrt{3}$*, then* $\frac{w_{\max}}{w_{\min}} \leq \frac{2\gamma^3}{1-3\gamma^2}$*. If* $\gamma \geq 1/\sqrt{3}$*, then* $\frac{w_{\max}}{w_{\min}}$ *can be unbounded.*

By combining Lemmas 3 and 4, we obtain the following result.

Theorem 7. *For* $\gamma < 1/\sqrt{3}$*, Algorithm 3 is a randomized* $\left(\frac{1}{2} + \frac{\gamma^3}{1-3\gamma^2} + \varepsilon\right)$*-approximation algorithm for* $\Delta(\gamma)$-ATSP*.*

We leave as an open problem to generalize the analysis to larger values of γ. However, it seems to be hard to find a constant factor approximation for $\gamma = 1$ since this would immediately yield a constant factor approximation for single-criterion Δ-ATSP

4.3 TSP with Weights One and Two

For both STSP$(1,2)$ and ATSP$(1,2)$, we have $\beta = 2$, i. e., $w_{\max}/w_{\min} = 2$. For STSP$(1,2)$, we have $\alpha \leq 1/3$, while we only have $\alpha \leq 1/2$ in case of ATSP$(1,2)$. The approximation ratio follows by exploiting Lemma 3. The edge weights and thus the objective functions are polynomially bounded for STSP$(1,2)$ and ATSP$(1,2)$. Thus, by Lemma 1, we can compute Pareto curves of cycle covers instead of only $(1+\varepsilon)$-approximate Pareto curves. Hence, we can get rid of the additional ε in the approximation ratios.

Theorem 8. *Algorithm 3 is a randomized* $4/3$ *approximation algorithm for multi-criteria* STSP$(1,2)$*.*

Theorem 9. *Algorithm 3 is a randomized* $3/2$ *approximation algorithm for multi-criteria* ATSP$(1,2)$*.*

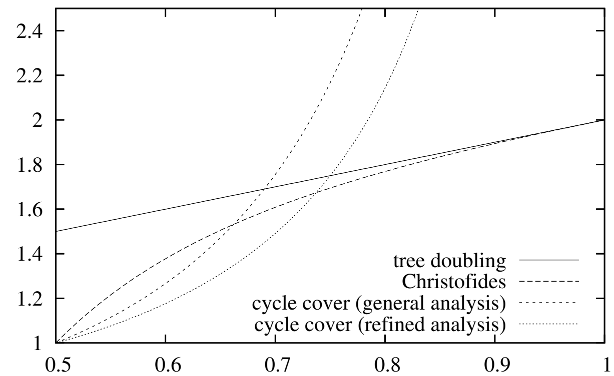


Fig. 2. Approximation ratios subject to γ achieved by tree doubling (Algorithm 1), Christofides' algorithm (Algorithm 2), and the cycle cover algorithm (Algorithm 3, Section 4), for which both the ratio obtained from the general and the refined analysis (Section 4.1) are shown.

5 Concluding Remarks

5.1 Comparing the Approximation Ratios for $\Delta(\gamma)$-STSP

Let us compare the approximation ratios achieved by the tree doubling algorithm (Section 2.1), Christofides' algorithm (Section 2.2), and the cycle cover algorithm (Section 4). Figure 2 shows the approximation ratios achieved by these algorithms subject to γ. Figure 3 shows the approximation ratios achieved deterministically (by the tree doubling algorithm) and randomized (by a combination of Christofides' and the cycle cover algorithm). The ratios are compared to the trivial ratio of $w_{\max}/w_{\min}$ and to the currently best known approximation ratio for single-criterion $\Delta(\gamma)$-STSP. Note that in particular for small values of γ, our algorithms for multi-criteria $\Delta(\gamma)$-STSP come close to achieving the ratio of the best algorithms for single-criterion $\Delta(\gamma)$-STSP.

5.2 Open Problems

Our approximation algorithm for multi-criteria $\Delta(\gamma)$-ATSP works only for $\gamma < 1/\sqrt{3}$. Thus, we are interested in finding constant factor approximation algorithms also for $\gamma \geq 1/\sqrt{3}$, which exist for all $\gamma < 1$ for single-criterion $\Delta(\gamma)$-ATSP [7, 9].

The cycle-cover-based algorithm for Max-TSP, where Hamiltonian cycles of maximum weight are sought, does not seem to perform well for multi-criteria Max-TSP. The reason for this is that the approximation algorithms for Max-TSP that base on cycle covers usually contain a statement like "remove the lightest edge of every cycle". While this works for single-criterion TSP, the term "lightest edge" is not well-defined for multi-criteria traveling salesman problems. We are particularly curious about the approximability of multi-criteria Max-TSP.

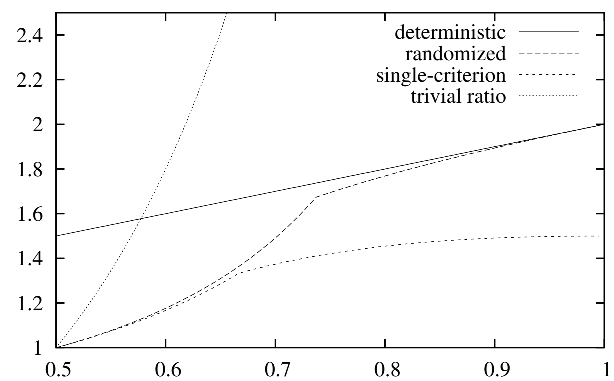


Fig. 3. Approximation ratios subject to γ. The deterministic ratio is achieved by tree doubling. Combining Christofides' and the cycle cover algorithm yields the randomized ratio. For comparison, the ratio for single-criterion $\Delta(\gamma)$-STSP and the trivial ratio $\frac{w_{\max}}{w_{\min}}$ are also shown.

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