Deterministic Algorithms for Multi-Criteria Max-TSP*

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We present deterministic approximation algorithms for the multi-criteria maximum traveling salesman problem (Max-TSP). Our algorithms are faster and simpler than the existing randomized algorithms.

We devise algorithms for the symmetric and asymmetric multi-criteria Max-TSP that achieve ratios of $1/2k - \varepsilon$ and $1/(4k-2) - \varepsilon$, respectively, where k is the number of objective functions. For two objective functions, we obtain ratios of $3/8 - \varepsilon$ and $1/4 - \varepsilon$ for the symmetric and asymmetric TSP, respectively. Our algorithms are self-contained and do not use existing approximation schemes as black boxes.

1 Multi-Criteria TSP

An instance of the traveling salesman problem (TSP) is a complete graph $G = (V, E)$ with edge weights $w : E \to \mathbb{Q}_+$. The goal is to find a *Hamiltonian cycle* (also called a *tour*) of minimum or maximum weight, where the weight of a tour is the sum of its edge weights. (The weight of an arbitrary set of edges is defined analogously.) If G is undirected, we have *Min-STSP* and *Max-STSP* (symmetric TSP). If G is directed, we have *Min-ATSP* and *Max-ATSP* (asymmetric TSP). For Min-ATSP and Min-STSP, we assume that the edge weights fulfill the triangle inequality, since otherwise the two problems cannot be approximated at all (assuming $P \neq NP$). All these variants of TSP are NP-hard and APX-hard [3]. Thus, we are in need of approximation algorithms. Table 1 shows the currently best approximation ratios for the four variants of the TSP.

In many scenarios, however, there is more than one objective function to optimize. In case of the TSP, we might want to minimize travel time, expenses, number of flight changes, etc., while we want to maximize, e.g., our profit along the route. This gives rise to multi-criteria TSP, where Hamiltonian cycles are sought that optimize several objectives simultaneously. In order to transfer the notion of optimal solutions to multi-criteria optimization problems, *Pareto curves* have been introduced (cf. Ehrgott [7]). A Pareto curve is a set of all optimal trade-offs between the different objective functions.

In the following, k always denotes the number of objective functions. We assume throughout the paper that $k \geq 2$ is an arbitrary constant. Let $[k] = \{1, 2, \ldots, k\}$. The k-criteria variants of

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the TSP that we consider are denoted by k-Min-STSP and k-Min-ATSP as well as k-Max-STSP and k-Max-ATSP.

We define the following terms for Max-TSP only. After that, we briefly point out the differences for Min-TSP. For a k-criteria variant of Max-TSP, we have edge weights $w_1, \ldots, w_k : E \to \mathbb{Q}_+$. For convenience, let $w = (w_1, \ldots, w_k)$. Inequalities of vectors are meant componentwise. A tour H *dominates* another tour $\tilde{H}$ if $w(H) \geq w(\tilde{H})$ and at least one of these k inequalities is strict. This means that H is strictly preferable to $\tilde{H}$. A *Pareto curve* is a set of all solutions that are not dominated by another solution. Since Pareto curves for the TSP cannot be computed efficiently, we have to be satisfied with approximate Pareto curves. A set $\mathcal{P}$ of tours is called an *α-approximate Pareto curve* for the instance (G, w) if the following holds: For every tour $\tilde{H}$ of G, there exists a tour $H \in \mathcal{P}$ of G with $w(H) \geq \alpha w(\tilde{H})$. We have $\alpha \leq 1$, and a 1-approximate Pareto curve is a Pareto curve. An algorithm is called an *α approximation algorithm* if it computes an α-approximate Pareto curve. A *fully polynomial time approximation scheme* (FPTAS) for a multi-criteria maximization problem computes $(1 - \varepsilon)$-approximate Pareto curves in time polynomial in the size of the instance and $1/\varepsilon$ for all $\varepsilon > 0$.

For Min-TSP, a tour H dominates $\tilde{H}$ if $w(H) \leq w(\tilde{H})$ and at least one inequality is strict. A set $\mathcal{P}$ of tours is an α-approximate Pareto curve if, for every tour $\tilde{H}$, we have an $H \in \mathcal{P}$ with $w(H) \leq \alpha w(\tilde{H})$. Note that $\alpha \geq 1$ for minimization problems. An FPTAS is a $(1 + \varepsilon)$ approximation algorithm.

1.1 Previous Work

Table 1 shows the current approximation ratios for the different variants of multi-criteria TSP. Many of these approximation algorithms can be extended to the case where some objectives should be minimized and others should be maximized [13]. We remark that an α approximation for Min-ATSP or Min-STSP yields a $k\alpha$ approximation for k-Min-ATSP or k-Min-STSP simply by encoding all objective functions into a single one. Thus, Feige and Singh's algorithm [8] yields a deterministic $\frac{2}{3} \cdot k \log_2 n$ approximation for k-Min-ATSP and Asadpour et al.'s algorithm [2] yields a randomized $O(k \frac{\log n}{\log \log n})$ approximation.

Unfortunately, no deterministic algorithms are known except for k-Min-STSP, k-Min-ATSP, and 2-Max-STSP. The reason for this is that most approximation algorithms for multi-criteria TSP use cycle covers. A *cycle cover* of a graph is a set of vertex-disjoint cycles such that every vertex is part of exactly one cycle. Hamiltonian cycles are special cases of cycle covers that consist of just one cycle. In contrast to Hamiltonian cycles, cycle covers of optimal weight can be computed in polynomial time. Cycle covers are among the main tools for designing approximation algorithms for the TSP [4–6, 8, 11, 17]. However, only a *randomized* fully polynomial-time approximation scheme (FPTAS) for multi-criteria cycle covers is known [19]. This randomized FPTAS builds on a reduction to a specific unweighted matching problem [18], which is then solved using the RNC algorithm by Mulmuley et al. [16]. Derandomizing this algorithm seems to be difficult [1], and these nested reductions make the algorithm quite slow. Hence, it is natural to ask whether there exist deterministic, faster approximation algorithms for multi-criteria TSP.

variant	*single-criterion* *randomized*	*deterministic*	*multi-criteria* *randomized*	*deterministic*	*new*
Min-STSP		3/2 [3]		$2+\varepsilon$ [15] 2 ($k=2$) [9]	
Min-ATSP	$O\big(\frac{\log n}{\log\log n}\big)$ [2]	$\frac{2}{3}\cdot\log_2 n$ [8]	$\log n+\varepsilon$ [14] $O\big(\frac{k\log n}{\log\log n}\big)$ [2]	$\frac{2}{3}\cdot k\log_2 n$ [8]	
Max-STSP		7/9 [17]	2/3 [10]	$\frac{7}{27}$ ($k=2$) [14]	$\frac{1}{2k}-\varepsilon$ $\frac{3}{8}-\varepsilon$ ($k=2$)
Max-ATSP		2/3 [11]	1/2 [10]		$\frac{1}{4k-2}-\varepsilon$ $\frac{1}{4}-\varepsilon$ ($k=2$)

Table 1: Approximation ratios for single-criterion and multi-criteria TSP.

1.2 New Results

We present deterministic approximation algorithms for multi-criteria Max-TSP, which are self-contained and considerably simpler and faster than the existing randomized algorithms. (Table 1 shows an overview.) Our algorithms do not use other algorithms as black boxes except for maximum-weight matching with a single objective function. Furthermore, they do not make any assumption about the representation of the edge weights. The existing algorithms require the (admittedly weak and natural) assumption that the edge weights are encoded in binary.

For k-Max-ATSP, we get a ratio of $\frac{1}{4k-2}-\varepsilon$ for any $\varepsilon>0$ (Section 2). For k-Max-STSP, we achieve a ratio of $\frac{1}{2k}-\varepsilon$ (Section 3). For the special case of two objective functions, we can improve this to $1/4-\varepsilon$ for 2-Max-ATSP and $3/8-\varepsilon$ for 2-Max-STSP. The latter is an improvement over the existing deterministic 7/27 approximation for 2-Max-STSP [14, 17].

2 Max-ATSP

The rough idea behind our algorithm for k-Max-ATSP is as follows: First, we "guess" a few edges that we contract to get a slightly smaller instance. The number of edges that we have to contract depends only on k and ε. Second, we compute k maximum-weight matchings in the smaller instance, each with respect to one of the k objective functions. Third, we compute another matching that uses only edges of the k matchings and that contains much weight with respect to each objective function. One note is here in order: Usually, cycle covers instead of matchings are used for Max-ATSP. However, although the weight of a cycle cover can be (roughly) twice as large as the weight of a maximum-weight matching, we do not get a better approximation ratio by using cycle covers. The reason is that we lose a factor of roughly 1/2 if we compute a collection of paths from k initial cycle covers compared to k initial matchings.

The following lemma is a key ingredient of our algorithm. It shows how to get a matching from k different matchings such that a significant fraction of the weight with respect to each matching is preserved. This works as long as no single edge contributes too much weight. The lemma immediately gives a polynomial-time algorithm for this task.

Lemma 2.1. *Let $G=(V,E)$ be a directed graph, and let $w=(w_1,\ldots,w_k)$ be edge weights. Let $M_1,\ldots,M_k\subseteq E$ be matchings. Let $\eta\in(0,1)$ be arbitrary such that $w_i(e)\leq\frac{\eta}{2k-2}\cdot w_i(M_i)$*

for all $e \in M_i$ and all $i \in [k]$. Then there exists a matching $P \subseteq \bigcup_{i=1}^k M_i$ such that $w_i(P) \geq \frac{1-\eta}{2k-1} \cdot w_i(M_i)$ for all $i \in [k]$. Such a matching P can be computed in polynomial time.

Proof. We construct the matching as follows: We add one heaviest edge $e \in M_1$ with respect to w_1 to P and remove e and all edges adjacent to e from $M_2, \ldots, M_k$. Then we put one heaviest remaining edge from M_2 into P and remove it and all adjacent edges. We proceed with $M_3, \ldots, M_k$ and repeat the process until no edges remain.

Let us analyze $w_i(P)$. In each step, at most two edges of any M_i are removed. Thus, we have removed at most $2i-2$ edges from M_i until we added the first edge from M_i to P. The weight of these edges is at most $(2i-2) \cdot \frac{\eta}{2k-2} w_i(M_i) \leq \eta w_i(M_i)$. Now let e be an edge of M_i that we added to P, and let $e_1, \ldots, e_t$ be the $t \leq 2k-2$ edges that are removed from M_i in the subsequent rounds of the procedure until again an edge of M_i is added. By construction, we have $w_i(e) \geq w_i(e_j)$ for all $j \in [t]$. Thus, $w_i(e) \geq \frac{1}{2k-1} \cdot (w_i(e) + \sum_{j=1}^t w_i(e_j))$. Taking the initial loss of $\eta w_i(M_i)$ into account, we observe that we can put a $\frac{1}{2k-1}$ fraction of $(1-\eta) w_i(M_i)$ into P for each $i \in [k]$. □

Now we have to make sure that, for a tour $\tilde{H}$, we can find appropriate matchings $M_1, \ldots, M_k$. For a directed complete graph $G = (V, E)$ and a set $K \subseteq E$ that forms a subset of a tour, we obtain G_{-K} by contracting all edges of K. Contracting an edge (u, v) means that we remove all outgoing edges of u and all incoming edges of v, and then identify u and v. We denote the vertex set of G_{-K} by V_{-K}. Analogously, for a tour $\tilde{H} \supseteq K$, we obtain a tour $\tilde{H}_{-K}$ by contracting the edges in K.

The following lemma says that, for any tour $\tilde{H}$, there is always a small set K of edges such that, if we contract these edges, the resulting tour $\tilde{H}_{-K}$ consists solely of edges that do not contribute too much to the weight of $\tilde{H}_{-K}$ with respect to any objective function. The proof is identical to the proof of the corresponding lemma for the $(1/2 - \varepsilon)$ approximation for k-Max-ATSP [13, 14]. In the algorithm, we will "guess" good sets K, compute Hamiltonian cycles on G_{-K}, and add the edges of K to get a Hamiltonian cycle of G. Small set means that $|K| \leq f(k, \varepsilon)$ for some function f that does not depend on the number n of vertices. We can choose $f(k, \varepsilon) \in O(k/\log(1/(1-\varepsilon))) = O(k/\log(1+\varepsilon)) = O(k/\varepsilon)$ [13, 14] (we have $1/\log(1+\varepsilon) = O(1/\varepsilon)$ by Taylor expansion). Moreover, we can choose K such that V_{-K} contains an even number of vertices. (Glaßer et al. [10] have proved a similar lemma with $|K| \in O(k)$. But their lemma does not provide a bound on the weight of the remaining edges.)

Lemma 2.2 (Manthey [14, Lemma 4.1])**.** *Let $G = (V, E)$ be a directed complete graph with edge weights $w = (w_1, \ldots, w_k)$, and let $\varepsilon > 0$. Let $H \subseteq E$ be any tour of G. Then there is a subset $K \subseteq H$ such that $|K| \leq f(k, \varepsilon)$ for some function $f(k, \varepsilon) = O(k/\varepsilon)$, $|V_{-K}|$ is even, and, for all $i \in [k]$, we have*

1. $w_i(K) \geq \frac{1}{4} \cdot w_i(H)$ *or*
2. $w_i(e) \leq \varepsilon \cdot w_i(H_{-K})$ *for all* $e \in H_{-K}$.

We have to make sure that any edge of G_{-K} weighs at most an ε fraction of $w(H_{-K})$, provided that $w(e) \leq \varepsilon w(H_{-K})$ for all $e \in H_{-K}$: Let $\beta_i = \max\{w_i(e) \mid e \in H_{-K}\}$ be the weight of the heaviest edge of H_{-K} with respect to w_i. Let $\beta = (\beta_1, \ldots, \beta_k)$. We define new edge weights w^β by setting the weight of edges that are too heavy with respect to some objective to 0:

$$w^\beta(e) = \begin{cases} w(e) & \text{if } w(e) \leq \beta \text{ and} \\ 0 & \text{if } w_i(e) > \beta_i \text{ for some } i. \end{cases}$$

$\mathcal{P}_{\text{TSP}} \leftarrow$ MaxATSP-Approx(G, w, ε)
input: directed complete graph $G = (V, E)$, $w : E \to \mathbb{Q}_+^k$, $\varepsilon > 0$
output: $(\frac{1}{4k-2} - \varepsilon)$-approximate Pareto curve $\mathcal{P}_{\text{TSP}}$ for k-Max-ATSP
for all $K \subseteq E$ that form a subset of a tour with $|K| \leq f(k, \varepsilon)$ and $|V_{-K}|$ even **do**
  for all $I \subseteq [k]$ and β **do**
    compute maximum-weight matchings M_i in G_{-K} w.r.t. w_i^β for $i \in \overline{I} = [k] \setminus I$
    compute a matching $P \subseteq \bigcup_{i \in \overline{I}} M_i$ according to Lemma 2.1
    add edges to $K \cup P$ to obtain a Hamiltonian cycle H; add H to $\mathcal{P}_{\text{TSP}}$

Algorithm 1: Approximation algorithm for k-Max-ATSP.

Since $w(e) \leq \beta$ for every $e \in H$ by definition, we have $w(H) = w^\beta(H)$. The number of vectors β that result in different weight functions w^β is bounded by n^{2k}: Since the number of edges is less than n^2, there are less than n^2 different edge weights for each objective function. Now we can state and analyze our approximation algorithm for k-Max-ATSP (Algorithm 1).

Theorem 2.3. *For every $\varepsilon > 0$ and $k \geq 2$, Algorithm 1 is a deterministic approximation algorithm for k-Max-ATSP that achieves an approximation ratio of $\frac{1}{4k-2} - \varepsilon$. Its running-time is $n^{O(k/\varepsilon)}$.*

Proof. We have to show that, for every tour $\tilde{H}$, there exists a tour $H \in \mathcal{P}_{\text{TSP}}$ with $w(H) \geq (\frac{1}{4k-2} - \varepsilon) \cdot w(\tilde{H})$. By Lemma 2.2, there exists a subset $K \subseteq \tilde{H}$ of edges and an $I \subseteq [k]$ such that $|K| \leq f(k, \varepsilon)$, $|V_{-K}|$ is even, $w_i(K) \geq w_i(\tilde{H})/4$ for all $i \in I$, and $w_i(e) \leq \varepsilon w_i(\tilde{H}_{-K})$ for all $e \in \tilde{H}_{-K}$ and $i \in [k] \setminus I$. Let $i \in [k] \setminus I$, and let β be defined by $\tilde{H}_{-K}$, i.e., $\beta_i = \max_{e \in \tilde{H}_{-K}} w_i(e)$. Consider the execution of the inner loop of Algorithm 1 corresponding to the considered values of K, I, and β, and let M_i for $i \in [k] \setminus I$ and P be the matchings constructed by Algorithm 1 in this execution of the inner loop. Note that M_i is a maximum-weight matching in G_{-K} with respect to w_i^β.

We have $w_i^\beta(M_i) \geq w_i^\beta(\tilde{H}_{-K})/2$, which implies

$$w_i^\beta(e) \leq w_i(e) \leq \varepsilon w_i(\tilde{H}_{-K}) = \varepsilon w_i^\beta(\tilde{H}_{-K}) \leq 2\varepsilon w_i^\beta(M_i)$$

for all $e \in M_i$. This together with Lemma 2.1 with $\eta = (2k-2)2\varepsilon$ yields

$$w_i^\beta(P) \geq \frac{1-\eta}{2k-1} \cdot w_i^\beta(M_i) = \frac{1-(2k-2)2\varepsilon}{2k-1} \cdot w_i^\beta(M_i) \geq \left(\frac{1}{2k-1} - 2\varepsilon\right) \cdot w_i^\beta(M_i).$$

The set $P \cup K$ of edges is a collection of paths in G. What remains to be done is to estimate the weight of $w(P \cup K)$. For every $i \in I$, we have $w_i(P \cup K) \geq w_i(K) \geq w_i(\tilde{H})/4 \geq \left(\frac{1}{4k-2} - \varepsilon\right) \cdot w_i(\tilde{H})$. For every $i \notin I$, we note that $w_i(\tilde{H}) = w_i(K) + w_i(\tilde{H}_{-K})$. This gives us

$$\begin{aligned} w_i(P \cup K) &\geq w_i^\beta(P) + w_i(K) \geq \left(\frac{1}{2k-1} - 2\varepsilon\right) \cdot w_i^\beta(M_i) + w_i(K) \\ &\geq \left(\frac{1}{4k-2} - \varepsilon\right) \cdot w_i^\beta(\tilde{H}_{-K}) + w_i(K) \geq \left(\frac{1}{4k-2} - \varepsilon\right) \cdot w_i(\tilde{H}). \end{aligned}$$

The running-time is bounded from above by $n^{O(1)+2k+f(k,\varepsilon)} = n^{O(k/\varepsilon)}$. □

If we have only two objective functions, we can improve the approximation ratio to $1/4 - \varepsilon$. The key ingredient for this is the following lemma, which is the improved counterpart of Lemma 2.1 for $k = 2$. The lemma can be proved using a cake-cutting argument with one player for each of the two objective functions.

Lemma 2.4. *Let $G = (V, E)$ be a directed graph with edge weights $w = (w_1, w_2)$ and an even number of vertices. Let $M_1, M_2 \subseteq E$ be two perfect matchings, and let $\eta \in (0, 1/4)$. Suppose that $w_i(e) \leq \frac{\eta}{2} \cdot w_i(M_i)$ for all $e \in M_i$ and $i \in \{1, 2\}$. Then there is a matching $P \subseteq M_1 \cup M_2$ with $w_i(P) \geq (\frac{1}{2} - \sqrt{\eta}) w_i(M_i)$ for $i \in \{1, 2\}$. The matching P can be found in polynomial time.*

Proof. Without loss of generality, we assume $M_1 \cap M_2 = \emptyset$. Otherwise, we can simply remove $M_1 \cap M_2$ from both matchings and add it to P. We scale the edge weights so that $w_i(M_i) = 1$ for $i \in \{1, 2\}$, and we will show how to obtain a matching $P \subseteq M_1 \cup M_2$ such that $w_1(P \cap M_1) \geq \frac{1}{2} - \sqrt{\eta}$ and $w_2(P \cap M_2) \geq \frac{1}{2} - \sqrt{\eta}$.

If we ignore the directions of the edges, the graph with edges $M_1 \cup M_2$ is a collection of disjoint cycles. Every cycle has even length and edges from M_1 and M_2 alternate.

Let $c \subseteq M_1 \cup M_2$ be a cycle. We say that c is a light cycle if $w_1(c) \leq \sqrt{\eta}$. Otherwise, i.e., if $w_1(c) > \sqrt{\eta}$, we call c a heavy cycle. Note that $M_1 \cup M_2$ has at most $1/\sqrt{\eta}$ heavy cycles.

We show the lemma by a cake-cutting argument: Player 1 puts cycles (or parts of cycles) into two sets S_1 and S_2, and then Player 2 can choose which set to take. Player i wants to maximize w_i. Player 1 puts light cycles as a whole into S_1 or S_2. Heavy cycles are split into two parts as follows: Player 1 decides to remove one edge of M_1 and one edge of M_2 (these edges are lost also for Player 2). In this way, we get two paths (again disregarding the directions of the edges). Player 1 puts one path into S_1 and the other path into S_2. (It can happen that one of the paths is empty: If we have a cycle of length four, the two edges removed are necessarily adjacent. This, however, does not cause any problem. In particular, cycles of length four are always light cycles.) Finally, Player 2 chooses the set S_i that maximizes w_2. Player 1 has to take S_{3-i}. This yields the matching $P = (S_i \cap M_2) \cup (S_{3-i} \cap M_1)$.

Let us estimate the weight that the players are guaranteed to get. Since we have at most $1/\sqrt{\eta}$ heavy cycles, at most $1/\sqrt{\eta}$ edges from M_2 are removed. The total weight of the edges removed is hence at most $\sqrt{\eta}/2$. Thus,

$$w_2\big((S_1 \cup S_2) \cap M_2\big) \geq w_2(M_2) - \sqrt{\eta}/2 = 1 - \sqrt{\eta}/2.$$

Hence, Player 2 can always get a weight of at least $\frac{1}{2} \cdot (1 - \sqrt{\eta}/2) \geq \frac{1}{2} - \sqrt{\eta}$.

Let us now focus on Player 1. As for Player 2, we have

$$w_1\big((S_1 \cup S_2) \cap M_1\big) \geq 1 - \sqrt{\eta}/2.$$

For any heavy weight cycle c, Player 1 can choose to remove edges such that the resulting paths differ by at most $\eta/2$ with respect to w_1. Since light cycles are put as a whole in either S_1 or S_2 and have a weight of at most $\sqrt{\eta}$ with respect to w_1, Player 1 can make sure that $w_1(S_1 \cap M_1)$ and $w_1(S_2 \cap M_1)$ differ by at most $\sqrt{\eta}$. Thus,

$$w_1(S_i \cap M_1) \geq \frac{1}{2} \cdot \left(1 - \frac{\sqrt{\eta}}{2}\right) - \frac{\sqrt{\eta}}{2} \geq \frac{1}{2} - \sqrt{\eta}$$

for both $i \in \{1, 2\}$. Thus, for any choice of Player 2, Player 1 still gets enough weight with respect to w_1. The proof immediately gives a polynomial-time algorithm for computing P. □

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𝒫_TSP ← MaxATSP-Approx-2(G, w, ε)
input: directed complete graph G = (V, E), w : E → ℚ²₊, ε > 0
output: (1/4 − ε)-approximate Pareto curve 𝒫_TSP for 2-Max-ATSP
for all K ⊆ E with |K| ≤ f(2, ε²) that are a subset of a tour and |V_{−K}| even do
    for all I ⊆ {1, 2} and β do
        compute maximum-weight matchings M_i in G_{−K} w.r.t. w_i^β for i ∈ Ī
        compute a matching P ⊆ ⋃_{i∈Ī} M_i according to Lemma 2.4
        add edges to K ∪ P to obtain a Hamiltonian cycle H; add H to 𝒫_TSP
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Algorithm 2: Improved approximation algorithm for 2-Max-ATSP.

Using Lemma 2.4, we obtain the following theorem.

Theorem 2.5. *For every $\varepsilon > 0$, Algorithm 2 is a deterministic approximation algorithm for 2-Max-ATSP with an approximation ratio of $1/4 - \varepsilon$. Its running-time is $n^{O(1/\varepsilon^2)}$.*

Proof. We have to prove that, for every tour $\tilde{H}$, there is an $H \in \mathcal{P}_{\mathrm{TSP}}$ with $w(H) \geq (\frac{1}{4} - \varepsilon) \cdot w(\tilde{H})$. According to Lemma 2.2, there is a subset $K \subseteq \tilde{H}$ and an $I \subseteq \{1, 2\}$ such that $|K| \leq f(2, \varepsilon^2)$, $|V_{-K}|$ is even, $w_i(K) \geq w_i(\tilde{H})/4$ for $i \in I$, and $w_i(e) \leq \varepsilon^2 w_i(\tilde{H}_{-K})$ for all $e \in \tilde{H}_{-K}$ and $i \in \{1, 2\} \setminus I = \overline{I}$. We choose $\beta = (\beta_1, \beta_2)$ with $\beta_i = \max_{e \in \tilde{H}_{-K}} w_i(e)$. Then $w_i^\beta(\tilde{H}_{-K}) = w_i(\tilde{H}_{-K})$ for all $i \in \overline{I}$.

We consider the execution of the inner loop of Algorithm 2 corresponding to the considered values of K, I, and β. Let P and M_i for $i \notin I$ be the corresponding matchings. Then $w_i^\beta(e) \leq 2\varepsilon^2 w_i^\beta(M_i)$ and $w_i^\beta(M_i) \geq \frac{1}{2} \cdot w_i^\beta(\tilde{H}_{-K})$.

Using Lemma 2.4 with $\eta = 4\varepsilon^2$, we have $w_i^\beta(P) \geq (\frac{1}{2} - 2\varepsilon) w_i^\beta(M_i)$ for each $i \in \overline{I}$. Again, $P \cup K$ is a collection of paths. For any $i \in I$, we have $w_i(P \cup K) \geq w_i(K) \geq w_i(\tilde{H})/4$, which is sufficient. For any $i \in \overline{I}$, we have

$$\begin{aligned} w_i(P \cup K) &\geq w_i^\beta(P) + w_i(K) \geq \left(\frac{1}{2} - 2\varepsilon\right) \cdot w_i^\beta(M_i) + w_i(K) \\ &\geq \left(\frac{1}{4} - \varepsilon\right) \cdot w_i^\beta(\tilde{H}_{-K}) + w_i(K) \geq \left(\frac{1}{4} - \varepsilon\right) \cdot w_i(\tilde{H}). \end{aligned}$$

The running-time is bounded by $n^{O(1) + f(2, 2\varepsilon^2)} = n^{O(1/\varepsilon^2)}$. □

3 Max-STSP

One key ingredient for our algorithm for k-Max-STSP is the following lemma, which is the undirected counterpart to Lemma 2.1. In contrast to k-Max-ATSP, we now start with k cycle covers rather than k matchings.

Lemma 3.1. *Let $G = (V, E)$ be an undirected graph with edge weights $w = (w_1, \ldots, w_k)$, and let $C_1, \ldots, C_k \subseteq E$ be cycle covers. Assume that, for some $\eta > 0$, we have $w_i(e) \leq \frac{\eta}{2k-1} w_i(C_i)$ for all $e \in C_i$ and all $i \in [k]$. Then there exists a collection $P \subseteq \bigcup_{i=1}^k C_i$ of vertex-disjoint paths such that $w_i(P) \geq \frac{1-\eta}{2k} w_i(C_i)$ for all i. Such a collection P can be computed in polynomial time.*

Proof. We have to select edges for P such that the degree of every vertex is at most two and that do not form any cycle. Instead of immediately removing edges (as in Lemma 2.1), we leave edges a "second chance": Only if two edges adjacent to an edge e are put into P, then we remove e. To keep track of which edges have to be removed, we *mark* an edge e if an edge adjacent to e is put into P. If e is already marked and another edge adjacent to e is put into P, then e is removed. The order in which we put the edges into P is as follows: We start with the heaviest edge with respect to w_1. Then we proceed with $w_2, \ldots, w_k$, then start over with w_1 again, and so on. If we proceed with some w_i, we select the heaviest edge with respect to w_i among all edges that are not yet removed.

Let $e \in C_i$ be an edge that we put into P, and let $e_1, \ldots, e_t \in \bigcup_{\ell=1}^{k} C_\ell$ be the edges that are adjacent to e. Then, different from the proof of Lemma 2.1, we do not remove $e_1, \ldots, e_t$, but we mark them. Only if an edge e_j is already marked, we remove it. Thus, an edge e_j is only removed if it is either put into P or if two edges adjacent to e are in P. Marked edges are still eligible for selection. This means that if we consider C_i, then we select the heaviest edge from C_i with respect to w_i that has not been deleted. Whether it is marked or not is irrelevant.

Now we claim that P is indeed a collection of paths. First, every vertex is incident to at most two edges of P: Assume that edges e and e' are adjacent to vertex v. If first e is added to P, all other edges adjacent to v (including e') are marked. If then e' is added to P, all edges adjacent to v are already marked. Thus, they will be deleted. This implies that P cannot contain a third edge incident to v.

Second, P does not contain cycles. Assume to the contrary that P contains a cycle. Let e be the last edge added to P, and let e' and e'' be the two edges of the cycle that are adjacent to e. If e' is added, then e is marked. If e'' is added afterwards, then e is deleted. Thus, e cannot be added to P, a contradiction.

Third, we have to prove that P contains enough weight with respect to $w_1, \ldots, w_k$. If an edge e is deleted, we charge half of this loss to the edge that caused e to be marked and half to the edge that caused e to be deleted.

Fix any i. At most $4i-2$ times, we have marked or removed an edge from C_i until we added the first edge from C_i to P (this includes the at most two edges of C_i that are marked while selecting an edge from C_i). The loss caused by these edges is at most $\frac{4i-2}{2} \cdot \frac{\eta}{2k-1} w_i(C_i) \leq \eta w_i(C_i)$ by assumption. Now let e be an edge of C_i that we add to P. Let $e_1, \ldots, e_t \in C_i$ be the $t \leq 4k-2$ edges of C_i that are removed or marked in this and the subsequent rounds of the procedure until again an edge of C_i is added (if an edge is both marked and removed during the subsequent rounds, it occurs twice on this list). By construction, we have $w_i(e) \geq w_i(e_j)$ for all $j \in [t]$. Thus, $w_i(e) \geq \frac{1}{2k} \cdot (w_i(e) + \frac{1}{2} \sum_{j=1}^{t} w_i(e_j))$. Taking into account the initial loss of at most $\eta w_i(C_i)$ yields $w_i(P) \geq \frac{1-\eta}{2k} \cdot w_i(C_i)$ for all $i \in [k]$. □

As in Section 2, we would like to keep a set $K \subseteq E$ of heavy edges. Unfortunately, it is impossible to contract edges in the same way as in directed graphs [14]. As already done for the randomized algorithms, we circumvent this by setting the weight along paths of sufficient length to 0 [13,14]. To do this formally, we need the following notation: Let $\tilde{H}$ be a Hamiltonian cycle, and let $K \subseteq \tilde{H}$. Let

$$L = L(K) = \{v \mid \exists e \in K : v \in e\}$$

be the set of vertices that are adjacent to edges of K. Let

$$T = T(K) = \{e \in \tilde{H} \mid e \text{ is adjacent to } K \text{ but not in } K\}.$$

$P \leftarrow$ MaxSTSP-Approx(G, w, ε)
input: undirected complete graph $G = (V, E)$, $w : E \to \mathbb{Q}_+^k$, $\varepsilon > 0$
output: $(\frac{1}{2k} - \varepsilon)$-approximate Pareto curve $\mathcal{P}_{\text{TSP}}$ for k-Max-STSP
for all $K \subseteq E$ with $|K| \leq g(k, \varepsilon/2)$ that form a subset of a tour **do**
  for all $I \subseteq [k]$, and β **do**
    compute maximum-weight cycle covers C_i in G w.r.t. $w_i^{-K,\beta}$ for $i \in \overline{I}$
    compute a collection $P \subseteq \bigcup_{i \in [k] \setminus I} C_i$ of paths according to Lemma 3.1
    remove edges incident to $L(K)$ from P to obtain P'
    add edges to $K \cup P'$ to obtain a Hamiltonian cycle H; add H to $\mathcal{P}_{\text{TSP}}$

Algorithm 3: $\frac{1}{2k} - \varepsilon$ approximation for k-Max-STSP.

As for the directed case, let $\beta = (\beta_1, \ldots, \beta_k)$. Now we define

$$w^{-L,\beta}(e) = \begin{cases} w(e) & \text{if } e \cap L = \emptyset \text{ and } w(e) \leq \beta \text{ and} \\ 0 & \text{if } e \cap L \neq \emptyset \text{ or there is an } i \text{ with } w_i(e) > \beta_i. \end{cases}$$

Furthermore, we define $w^{-K,\beta} = w^{-L(K),\beta}$. This means that under $w^{-K,\beta}$, all edges of K or adjacent to K have weight 0. Furthermore, all edges that exceed β for some objective are also set to 0.

If we omit the parameter β, then no edges are set to 0 because they are too heavy, i.e., $w^{-L} = w^{-L,(\infty,\ldots,\infty)}$ and $w^{-K} = w^{-K,(\infty,\ldots,\infty)}$.

Now we are prepared to state the undirected counterpart of Lemma 2.2. As in Lemma 2.2, its proof is identical to the proof of the corresponding lemma for the $(\frac{2}{3} - \varepsilon)$ approximation for k-Max-STSP [13, 14]. We can choose the function g in the lemma such that $g(k, \eta) \in O\big(\frac{k^3}{\eta \cdot (\log(1-\eta))^2}\big) = O(k^3/\eta^3)$.

In the following, we assume that $|V|$ is even. If $|V|$ is indeed odd, then only the analysis becomes a bit more technical, but the decrease of the approximation ratio is negligible.

Lemma 3.2 (Manthey [14, Lemma 4.5])**.** *Let $G = (V, E)$ be an undirected complete graph, and let $w = (w_1, \ldots, w_k)$ be edge weights. Let $\eta > 0$. Let $H \subseteq E$ be any Hamiltonian cycle of G. Then there exists a collection $K \subseteq H$ of paths such that $|K| \leq g(k, \eta)$ for some function $g(k, \varepsilon) = O(k^3/\eta^2)$ and the following properties hold: Let $L = L(K)$ and $T = T(K)$. For all $i \in [k]$, we have*

1. *$w_i(K) \geq \frac{1}{2} \cdot w_i(H)$ or*
2. *$w_i(e) \leq \eta \cdot w_i^{-K}(H)$ for all $e \in H \setminus K$ and $w_i(T) \leq \eta \cdot w_i(H)$.*

Now we are prepared to state and analyze our approximation algorithm for k-Max-STSP (Algorithm 3), and we obtain the following theorem.

Theorem 3.3. *For every $k \geq 2$ and $\varepsilon > 0$, Algorithm 3 is a deterministic approximation algorithm for k-Max-STSP that achieves an approximation ratio of $\frac{1}{2k} - \varepsilon$ and has running-time $n^{O(k^3/\varepsilon^3)}$.*

Proof. We have to show that, for every Hamiltonian cycle $\tilde{H}$, there exists a Hamiltonian cycle $H \in \mathcal{P}_{\text{TSP}}$ with $w(H) \geq (\frac{1}{2k} - \varepsilon) \cdot w(\tilde{H})$. By Lemma 3.2 with $\eta = \varepsilon/2$, there exists a subset $K \subseteq \tilde{H}$ of edges and an $I \subseteq [k]$ such that $|K| \leq g(k, \varepsilon/2)$ and the following properties are met:

1. For all $i \in I$, we have $w_i(K) \geq w_i(\tilde{H})/2$.
2. For all $i \in [k] \setminus I = \overline{I}$, we have $w_i(e) \leq \frac{\varepsilon}{2} \cdot w_i^{-K}(\tilde{H})$ for all $e \in \tilde{H} \setminus K$ and $w_i(T) \leq \frac{\varepsilon}{2} \cdot w_i(\tilde{H})$.

Let $\beta = (\beta_1, \ldots, \beta_k)$ with $\beta_i = \max_{e \in \tilde{H}} w_i^{-K}(e)$. For all $i \in \overline{I}$, the following properties hold for the maximum-weight cycle covers C_i with respect to $w_i^{-K,\beta}$:

1. $w_i^{-K,\beta}(C_i) \geq w_i^{-K,\beta}(\tilde{H})$ and
2. $w_i^{-K,\beta}(e) \leq \frac{\varepsilon}{2} \cdot w_i^{-K,\beta}(C_i)$ for all $e \in C_i$.

The first property holds since the cycle cover weight bounds the weight of the Hamiltonian cycle from above. The second property holds because of the following sequence of inequalities:

$$w_i^{-K,\beta}(e) \leq \frac{\varepsilon}{2} \cdot w_i^{-K}(\tilde{H}) = \frac{\varepsilon}{2} \cdot w_i^{-K,\beta}(\tilde{H}) \leq \frac{\varepsilon}{2} \cdot w_i^{-K,\beta}(C_i).$$

Now we consider the execution of the inner loop of Algorithm 3 for the corresponding K, I, and β. Let C_i for $i \in [k] \setminus I$ be the corresponding cycle covers, and let P be the corresponding collection of paths. Lemma 3.1 with $\eta = k\varepsilon$ shows that

$$w_i^{-K,\beta}(P) \geq \frac{1 - k\varepsilon}{2k} w_i^{-K,\beta}(C_i).$$

Without changing the weight of P with respect to $w^{-K,\beta}$, we can remove all edges incident to $L = L(K)$ from P. Let $P' \subseteq P$ be the corresponding subset constructed in Line 5 of Algorithm 3. The set $P' \cup K$ is a collection of paths in G, and we have $w^{-K,\beta}(P') = w^{-K,\beta}(P)$. What remains to be done is to estimate the weight of $w(P' \cup K)$. For every $i \in I$, we have

$$w_i(P' \cup K) \geq w_i(K) \geq \frac{w_i(\tilde{H})}{2} \geq \left(\frac{1}{2k} - \varepsilon\right) \cdot w_i(\tilde{H}).$$

For any $i \notin I$, we observe that $w_i(\tilde{H}) = w_i(K) + w_i^{-K,\beta}(\tilde{H} \setminus K) + w_i(T)$. Recalling that $w_i(T) \leq \varepsilon w_i(\tilde{H})$ yields

$$\begin{aligned} w_i(P' \cup K) &\geq w_i^{-K,\beta}(P) + w_i(K) \geq \frac{1 - k\varepsilon}{2k} \cdot w_i^{-K,\beta}(C_i) + w_i(K) \\ &\geq \frac{1 - k\varepsilon}{2k} \cdot w_i^{-K,\beta}(\tilde{H} \setminus K) + w_i(K) \\ &\geq \left(\frac{1}{2k} - \frac{\varepsilon}{2}\right) \cdot \left(w_i^{-K,\beta}(\tilde{H} \setminus K) + w_i(K)\right) \\ &\geq (1 - \varepsilon) \cdot \left(\frac{1}{2k} - \frac{\varepsilon}{2}\right) \cdot w_i(\tilde{H}) \geq \left(\frac{1}{2k} - \varepsilon\right) \cdot w_i(\tilde{H}). \end{aligned}$$

The bound on the running-time follows from $g(k, \varepsilon/2) \in O(k^3/\varepsilon^3)$. □

As for 2-Max-ATSP, we can achieve a better approximation ratio of $3/8 - \varepsilon$ for $k = 2$. This improves over the known deterministic $7/27$ approximation [14, 17].

Lemma 3.4. *Let $G = (V, E)$ be an undirected graph with edge weights $w = (w_1, w_2)$, and let $M_1, M_2 \subseteq E$ be two perfect matchings. Assume that $w_i(e) \leq \eta w_i(M_i)$ for $i \in \{1, 2\}$ and all edges $e \in M_i$. Then there exists a collection $P \subseteq M_1 \cup M_2$ of paths such that $w_i(P) \geq (\frac{3}{4} - \eta) \cdot w_i(M_i)$ for $i \in \{1, 2\}$. Such a collection P can be found in polynomial time.*

$\mathcal{P}_{\text{TSP}} \leftarrow$ MaxSTSP-Approx-2(G, w, ε)

input: undirected complete graph $G = (V, E)$, $w : E \to \mathbb{Q}_+^2$, $\varepsilon > 0$

output: $(\frac{3}{8} - \varepsilon)$-approximate Pareto curve $\mathcal{P}_{\text{TSP}}$ for 2-Max-STSP

for all $K \subseteq E$ with $|K| \le g(2, \varepsilon/2)$ that form a subset of a tour **do**

  for all $I \subseteq \{1, 2\}$ and β **do**

    compute maximum-weight matchings M_i in G w.r.t. $w_i^{-K,\beta}$ for $i \in \overline{I}$

    compute a collection $P \subseteq \bigcup_{i \in [k] \setminus I} M_i$ of paths according to Lemma 3.4

    remove edges incident to $L(K)$ from P to obtain P'

    add edges to $K \cup P'$ to obtain a Hamiltonian cycle H; add H to $\mathcal{P}_{\text{TSP}}$

Algorithm 4: Improved approximation for 2-Max-STSP.

Proof. Without loss of generality, we assume $M_1 \cap M_2 = \emptyset$. Otherwise, we can simply remove $M_1 \cap M_2$ from both matchings and add it to P. Thus, $M_1 \cup M_2$ forms a graph consisting solely of simple cycles of even length. Every cycle of $M_1 \cup M_2$ has a length of at least 4.

For every cycle c of length at least eight, we can simply remove either the lightest edge of $c \cap M_1$ with respect to w_1 or the lightest edge of $c \cap M_2$ with respect to w_2. In this way, at least $\frac{3}{4} \cdot w(c)$ of the weight of c is preserved.

Thus, it remains to deal with the shorter cycles. These cycles have a length of four or six. We deal with them by a cake-cutting argument: First, Player 1 partitions the cycles into two sets C_1 and C_2. Second, Player 2 chooses some C_i. Finally, the lightest edge of M_1 is removed from any cycle in C_i, and the lightest edge of M_2 is removed from any cycle in C_{3-i}. Thus, Player 1 gets at least half the weight of C_i plus the full weight of C_{3-i}. Analogously, Player 2 gets the full weight of C_i plus at least half the weight of C_{3-i}. The paths from the long cycles together with the paths obtained from C_1 and C_2 yields a collection of paths.

Now, let W_1 and W_2 be the weight of M_1 and M_2, respectively, that is contained in cycles with a length of at least eight. Player 2's goal is to maximize w_2. If Player 2 chooses C_i, then $w_2(P) \ge \frac{3}{4} \cdot W_2 + w_2(C_i) + \frac{1}{2} \cdot w_2(C_{3-i})$. Furthermore, we have $w_2(C_i) + w_2(C_{3-i}) + W_2 = w_2(M_2)$. Thus, Player 2 can always achieve $w_2(P) \ge \frac{3}{4} \cdot w_2(M_2)$, independent of how Player 1 constructs C_1 and C_2.

Let us now focus on Player 1, who wants to maximize w_1. Player 1 divides the cycles into C_1 and C_2 such that $w_1(C_1 \cap M_1)$ and $w_1(C_2 \cap M_1)$ differ by at most $3\eta w_1(M_1)$. This can easily be achieved because any cycle contains at most three edges of M_1 and $w_1(e) \le \eta w_1(M_1)$ for all $e \in M_1$.

Now we assume that Player 2 chooses C_i. We have $w_1(C_i \cap M_1) + w_1(C_{3-i} \cap M_1) = w_1(M_1) - W_1$ and $w_1(C_i \cap M_1) - w_1(C_{3-i} \cap M_1) \le 3\eta w_1(M_1)$. Player 1 loses at most $w_1(C_i)/2$ due to Player 2 removing edges of M_1 from C_i. Thus, we have

$$\begin{aligned} w_1(P) &\ge \frac{3}{4} \cdot W_1 + \left(w_1(C_{3-i} \cap M_1) + \frac{1}{2} \cdot w_1(C_i \cap M_1) \right) \\ &\ge \frac{3}{4} \cdot W_1 + \left(\frac{3}{4} \cdot w_1(C_{3-i} \cap M_1) + \frac{3}{4} \cdot w_1(C_i \cap M_1) - \frac{1}{4} \cdot 3\eta w_1(M_1) \right) \\ &\ge \left(\frac{3}{4} - \frac{3\eta}{4} \right) \cdot w_1(M_1), \end{aligned}$$

which is enough. The proof directly yields a polynomial-time algorithm for computing P. □

Theorem 3.5. *For any $\varepsilon > 0$, Algorithm 4 is a deterministic algorithm for 2-Max-STSP with an approximation ratio of $\frac{3}{8} - \varepsilon$. Its running-time is $n^{O(1/\varepsilon^3)}$.*

Proof. We have to prove that, for every Hamiltonian cycle $\tilde{H}$, there is an $H \in \mathcal{P}_{\text{TSP}}$ with $w(H) \geq (\frac{3}{8} - \varepsilon) \cdot w(\tilde{H})$. According to Lemma 3.2 with $\eta = \varepsilon/2$, there exist $K \subseteq \tilde{H}$ and $I \subseteq \{1, 2\}$ such that $|K| \leq g(k, \varepsilon/2)$ and

1. $w_i(K) \geq w_i(\tilde{H})/2$ for $i \in I$ and
2. $w_i^{-K}(e) \leq \frac{\varepsilon}{2} \cdot w_i(\tilde{H} \setminus K)$ for all $e \in \tilde{H} \setminus K$ and $w_i(T) \leq \frac{\varepsilon}{2} \cdot w_i(\tilde{H})$ for all $i \in \{1, 2\} \setminus I = \overline{I}$.

Let $\beta = (\beta_1, \beta_2)$ with $\beta_i = \max_{e \in \tilde{H}} w_i^{-K}(\tilde{H})$. Then $w_i^{-K,\beta}(\tilde{H} \setminus K) = w_i^{-K}(\tilde{H} \setminus K)$ for all $i \in \overline{I}$. Consider the execution of the inner loop of Algorithm 4 for the corresponding values of K, I, and β. Let P be the corresponding set of paths obtained from the matchings M_i for $i \in [2] \setminus I$. Then we have

1. $w_i^{-K,\beta}(M_i) \geq w_i^{-K,\beta}(\tilde{H} \setminus K)/2$ and
2. $w_i^{-K,\beta}(e) \leq \varepsilon \cdot w_i^{-K,\beta}(M_i)$ for all $e \in M_i$.

Applying Lemma 3.4 with $\eta = \varepsilon$ yields $w_i^{-K,\beta}(P) \geq (\frac{3}{4} - \varepsilon) \cdot w_i^{-K,\beta}(M_i)$ for each $i \in \overline{I}$. We remove all edges incident to $L(K)$ from P to obtain P'. By construction, we have $w^{-K,\beta}(P') = w^{-K,\beta}(P)$.

The set $P' \cup K$ of edges is a collection of paths. For any $i \in I$, we have $w_i(P \cup K) \geq w_i(K) \geq w_i(\tilde{H})/2$. We observe that $w(\tilde{H}) = w^{-K,\beta}(\tilde{H}) + w(K) + w(T)$ and $w_i(T) \leq \frac{\varepsilon}{2} \cdot w_i(\tilde{H})$ for any $i \in \overline{I}$. Thus, for any $i \in \overline{I}$, we have

$$
\begin{aligned}
w_i(P' \cup K) &\geq w_i^{-K,\beta}(P) + w_i(K) \geq \left(\frac{3}{4} - \varepsilon\right) \cdot w_i^{-K,\beta}(M_i) + w_i(K) \\
&\geq \left(\frac{3}{8} - \frac{\varepsilon}{2}\right) \cdot w_i^{-K,\beta}(\tilde{H}) + w_i(K) \\
&\geq \left(1 - \frac{\varepsilon}{2}\right) \cdot \left(\frac{3}{8} - \frac{\varepsilon}{2}\right) \cdot w_i(\tilde{H}) \geq \left(\frac{3}{8} - \varepsilon\right) \cdot w_i(\tilde{H}).
\end{aligned}
$$

Finally, the bound on the running-time follows from $g(2, \varepsilon/2) \in O(1/\varepsilon^3)$. □

4 Open Problems

We conclude with three open questions: First, does there exist a *deterministic* approximation algorithm for k-Min-ATSP with a non-trivial approximation ratio? Non-trivial means smaller than $k \cdot \frac{2}{3} \cdot \log_2 n$, which is obtained by adding the k weights of each edge to get a single objective function. (Such trivial approximation algorithms do not exist for maximization problems.) A key step towards this goal would be an approximation scheme for multi-criteria *perfect* matching. However, a derandomization of the randomized FPTAS for general matching [19], which is based on the isolation lemma [16], seems to be difficult [1].

Second, are there *deterministic* approximation algorithms for k-Max-ATSP and k-Max-STSP that achieve constant approximation ratios (or at least ratios of $\omega(1/k)$)?

Third, are there *deterministic* algorithms for the case where some objectives should be minimized while others should be maximized?

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