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# On Approximating Multi-Criteria TSP

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We present approximation algorithms for almost all variants of the multi-criteria traveling salesman problem (TSP).

First, we devise randomized approximation algorithms for multi-criteria maximum traveling salesman problems (Max-TSP). For multi-criteria Max-STSP, where the edge weights have to be symmetric, we devise an algorithm with an approximation ratio of $2/3 - \varepsilon$. For multi-criteria Max-ATSP, where the edge weights may be asymmetric, we present an algorithm with a ratio of $1/2 - \varepsilon$. Our algorithms work for any fixed number $k$ of objectives. Furthermore, we present a deterministic algorithm for bi-criteria Max-STSP that achieves an approximation ratio of $7/27$.

Finally, we present a randomized approximation algorithm for the asymmetric multi-criteria minimum TSP with triangle inequality (Min-ATSP). This algorithm achieves a ratio of $\log n + \varepsilon$.

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## 1. MULTI-CRITERIA TRAVELING SALESMAN PROBLEM

### 1.1. Traveling Salesman Problem

The traveling salesman problem (TSP) is one of the most famous combinatorial optimization problems. Given a graph, the goal is to find a Hamiltonian cycle of maximum or minimum weight (Max-TSP or Min-TSP). An instance of *Max-TSP* is a complete graph $G = (V, E)$ with edge weights $w : E \to \mathbb{Q}_+$. The goal is to find a Hamiltonian cycle of maximum weight. The weight of a Hamiltonian cycle (more general, of any set of edges) is the sum of the weights of its edges. If $G$ is undirected, we have *Max-STSP* (symmetric TSP). If $G$ is directed, we obtain *Max-ATSP* (asymmetric TSP).

An instance of *Min-TSP* is also a complete graph $G$ with edge weights $w$ that fulfill the triangle inequality: $w(u, v) \leq w(u, x) + w(x, v)$ for all $u, v, x \in V$. The goal is to find a Hamiltonian cycle of minimum weight. We have *Min-STSP* if $G$ is undirected and *Min-ATSP* if $G$ is directed.

All these variants are NP-hard and APX-hard. Thus, we have to content ourselves with approximate solutions. The currently best approximation algorithm for Max-STSP achieves

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an approximation ratio of 7/9 [Paluch et al. 2009], and the currently best algorithm for Max-ATSP achieves a ratio of 2/3 [Kaplan et al. 2005]. Min-ATSP can be approximated with a factor of $O(\log n/\log\log n)$, where $n$ is the number of vertices of the instance [Asadpour et al. 2010].

Cycle covers are often used for designing approximation algorithms for the TSP [Bläser and Manthey 2005; Kaplan et al. 2005; Feige and Singh 2007]. A *cycle cover* of a graph is a set of vertex-disjoint cycles such that every vertex is part of exactly one cycle. The general idea is to compute an initial cycle cover and then to join the cycles to obtain a Hamiltonian cycle. This technique is called *subtour patching* [Gilmore et al. 1985]. Hamiltonian cycles are special cases of cycle covers that consist of a single cycle. Thus, the weight of a maximum-weight cycle cover bounds the weight of a maximum-weight Hamiltonian cycle from above, and minimum-weight cycle covers provide lower bounds for minimum-weight Hamiltonian cycles. In contrast to Hamiltonian cycles, cycle covers of maximum or minimum weight can be computed efficiently by reduction to matching problems [Ahuja et al. 1993].

### 1.2. Multi-Criteria Optimization

In many optimization problems, there is more than one objective function. This is also the case for the TSP: We might want to minimize travel time, expenses, number of flight changes, etc., while maximizing, e.g., our profit along the way. This leads to $k$-criteria variants of the TSP (*$k$-Max-STSP*, *$k$-Max-ATSP*, *$k$-Min-STSP*, and *$k$-Min-ATSP* for short).

With respect to a single criterion, the term “optimal solution” is well-defined. However, if several criteria are involved, there is no natural notion of a best choice. Instead, we have to be satisfied with trade-off solutions. The goal of multi-criteria optimization is to cope with this dilemma. To transfer the concept of optimal solutions to multi-criteria optimization problems, the notion of *Pareto curves* (also known as *Pareto sets* or *efficient sets*) has been introduced [Ehrgott 2005]. A Pareto curve is a set of solutions that can be considered optimal.

We introduce the following terms only for maximization problems. After that, we briefly state the differences for minimization problems. An instance of $k$-Max-TSP is a complete graph $G$ with edge weights $w_1, \ldots, w_k : E \to \mathbb{Q}_+$. A Hamiltonian cycle $H$ *dominates* another Hamiltonian cycle $\tilde{H}$ if $w_i(H) \geq w_i(\tilde{H})$ for all $i \in [k] = \{1, \ldots, k\}$ and $w_i(H) > w_i(\tilde{H})$ for at least one $i$. This means that $H$ is strictly preferable to $\tilde{H}$. A *Pareto curve* of solutions contains all solutions that are not dominated by another solution. For other maximization problems, $k$-criteria variants are defined analogously.

Unfortunately, Pareto curves cannot be computed efficiently in many cases: First, they are often of exponential size. Second, because of reductions from knapsack problems, they are NP-hard to compute even for otherwise easy optimization problems. Third, TSP is NP-hard already with only one objective function, and optimization problems do not become easier with more objectives involved. Therefore, we have to be satisfied with approximate Pareto curves.

For simpler notation, let $w(H) = (w_1(H), \ldots, w_k(H))$. Inequalities are meant component-wise. A set $\mathcal{P}$ of Hamiltonian cycles of $V$ is called an *$\alpha$ approximate Pareto curve* for $(G, w)$ if the following holds: For every Hamiltonian cycle $\tilde{H}$, there exists a Hamiltonian cycle $H \in \mathcal{P}$ with $w(H) \geq \alpha w(\tilde{H})$. We have $\alpha \leq 1$, and a 1 approximate Pareto curve is a Pareto curve.

An algorithm is called an *$\alpha$ approximation algorithm* if, given $G$ and $w$, it computes an $\alpha$ approximate Pareto curve. It is called a *randomized $\alpha$ approximation* if its success probability is at least 1/2. This success probability can be amplified to $1-2^{-m}$ by executing the algorithm $m$ times and taking the union of all sets of solutions. (We can also remove solutions from this union that are dominated by other solutions in the union.) Again, the concepts can be transfered easily to other maximization problems.

Papadimitriou and Yannakakis [2000] have shown that $(1-\varepsilon)$ approximate Pareto curves of size polynomial in the instance size and $1/\varepsilon$ exist. The technical requirement for the existence is that the objective values of all solutions for an instance $X$ are either 0 or within an interval $[2^{-p(N)}, 2^{p(N)}]$ for some polynomial $p$, where $N$ is the size of $X$. This is fulfilled by most optimization problems and in particular in our case.

A *fully polynomial time approximation scheme* (FPTAS) for a multi-criteria optimization problem computes $(1-\varepsilon)$ approximate Pareto curves in time polynomial in the size of the instance and $1/\varepsilon$ for all $\varepsilon > 0$. Multi-criteria maximum-weight matching admits a *randomized FPTAS* [Papadimitriou and Yannakakis 2000], i. e., the algorithm succeeds in computing a $(1-\varepsilon)$ approximate Pareto curve with a probability of at least 1/2. This randomized FPTAS yields also a randomized FPTAS for the multi-criteria maximum-weight cycle cover problem [Manthey and Ram 2009].

To define Pareto curves and approximate Pareto curves also for minimization problems, in particular for $k$-Min-STSP and $k$-Min-ATSP, we have to replace all "$\geq$" and "$>$" above by "$\leq$" and "$<$". Furthermore, $\alpha$ approximate Pareto curves are now defined for $\alpha \geq 1$, and an FPTAS achieves an approximation ratio of $1+\varepsilon$. There also exists a randomized FPTAS for the multi-criteria minimum-weight cycle cover problem.

### 1.3. Known Results

For an overview of the literature about multi-criteria optimization, including multi-criteria TSP, we refer to Ehrgott and Gandibleux [2000; 2005].

Angel et al. [2004; 2005] have considered $k$-Min-STSP restricted to edge weights 1 and 2. They analyzed a local search heuristic and proved that it achieves an approximation ratio of 3/2 for $k = 2$ and of $2 - \frac{2}{k+1}$ for $k \geq 3$. Ehrgott [2000] has analyzed a variant of $k$-Min-STSP, where all objectives are encoded into a single objective by using some norm. He proved approximation ratios between 3/2 and 2 for this problem, where the ratio depends on the norm used. $k$-Min-STSP can be approximated with a ratio of $2+\varepsilon$ [Manthey and Ram 2009]. For $k$-Min-ATSP, we are not aware of any prior approximation algorithm.

Bläser et al. [2008] have devised the first randomized approximations for $k$-Max-STSP and $k$-Max-ATSP. Their algorithms achieve ratios of $\frac{1}{k} - \varepsilon$ for $k$-Max-STSP and $\frac{1}{k+1} - \varepsilon$ for $k$-Max-ATSP. They have conjectured that approximation ratios of $\Omega(1/\log k)$ are possible.

### 1.4. New Results

We devise approximation algorithms for $k$-Max-STSP, $k$-Max-ATSP, and $k$-Min-ATSP that work for any number $k$ of criteria.

First, we solve the conjecture of Bläser et al. [2008] affirmatively. We even prove a stronger result: For $k$-Max-STSP, we achieve a ratio of $2/3 - \varepsilon$, while for $k$-Max-ATSP, we achieve a ratio of $1/2 - \varepsilon$ (Section 4). The general idea of these algorithms is sketched in Section 2. After that, we introduce a decomposition technique in Section 3 that will lead to our algorithms (Section 4). Our algorithms are randomized and their running-time is polynomial in the input size for any fixed $\varepsilon > 0$ and any fixed number $k$ of criteria.

Furthermore, as a first step towards deterministic approximation algorithms for $k$-Max-TSP, we devise an approximation algorithm for 2-Max-STSP that achieves an approximation ratio of 7/27 (Section 5). As a side effect, this result proves that for 2-Max-STSP, there always exists a single Hamiltonian cycle that already is a 1/3 approximate Pareto curve. This does not hold for any other variant of multi-criteria TSP.

Finally, we devise the first approximation algorithm for $k$-Min-ATSP (Section 6). The approximation ratio of our algorithm is $\log n + \varepsilon$, where $n$ is the number of vertices. Our algorithm is randomized and its running-time is polynomial in the input size and in $1/\varepsilon$ for any fixed number of criteria.

## 2. IDEA FOR MULTI-CRITERIA MAX-TSP

For Max-ATSP, we can easily get a $1/2$ approximation: We compute a maximum-weight cycle cover and remove the lightest edge of each cycle. This yields a collection of paths. Then we add edges to connect the paths, which yields a Hamiltonian cycle. For Max-STSP, this approach gives a ratio of $2/3$ since the length of every cycle is at least three.

Unfortunately, this does not generalize to multi-criteria Max-TSP. The reason is that the term "lightest edge" is not well defined: An edge that has little weight with respect to one objective might have a huge weight with respect to another objective. Based on this observation, the basic idea behind our algorithms is "guessing" the heavy edges such that the remaining edges are all light-weight. A similar technique has already been used by Ravi and Goemans [1996] for bi-criteria spanning trees. Since the remaining edges are light-weight, and we can break one edge of every cycle without losing too much weight with respect to any objective function. This is based on the decompositions introduced in the following section.

## 3. DECOMPOSITIONS

Given a cycle cover $C$, a decomposition of $C$ is a collection $P \subseteq C$ of paths. From such a collection $P$, we obtain a Hamiltonian cycle just by connecting the endpoints of the paths appropriately. In order to get approximation algorithms for multi-criteria Max-TSP, our goal is to find collections $P$ with $w(P) \geq \alpha w(C)$ for an $\alpha$ as large as possible. Decompositions have already been used by Bläser et al. [2008] for their approximation algorithms for multi-criteria Max-TSP. With their decompositions, they have achieved approximation ratios of $\frac{1}{k} - \varepsilon$ and $\frac{1}{k+1} - \varepsilon$, and they conjectured that approximation ratio $\Omega(1/\log k)$ is possible. We introduce a slightly different kind of decompositions, which enables us to design constant-factor approximations.

Let $C$ be a cycle cover, and let $w = (w_1, \ldots, w_k)$ be edge weights. We say that the pair $(C, w)$ is $\eta$-light for some $\eta \leq 1$ if $w(e) \leq \eta w(C)$ for all $e \in C$. From now on, let $\eta_{k,\varepsilon} = \frac{\varepsilon^2}{2 \ln k}$.

THEOREM 3.1. *Let $\varepsilon \in (0, \frac{1}{2})$ be arbitrary, and let $k \geq 2$ be arbitrary. Let $C$ be a cycle cover, and let $w = (w_1, \ldots, w_k)$ be edge weights such that $(C, w)$ is $\eta_{k,\varepsilon}$-light.*

*If $C$ is directed, then there exists a collection $P \subseteq C$ of paths with $w(P) \geq (\frac{1}{2} - \varepsilon) \cdot w(C)$.*

*If $C$ is undirected, then there exists a collection $P \subseteq C$ of paths with $w(P) \geq (\frac{2}{3} - \varepsilon) \cdot w(C)$.*

PROOF. The proof uses Hoeffding's inequality [1963, Theorem 2].

LEMMA 3.2 (HOEFFDING'S INEQUALITY). *Let $X_1, \ldots, X_n$ be independent random variables, where $X_j$ assumes values in the interval $[a_j, b_j]$. Let $X = \sum_{j=1}^{n} X_j$. Then*

$$\mathbb{P}\big(X < \mathbb{E}(X) - t\big) \leq \exp\left(-\frac{2t^2}{\sum_{j=1}^{n}(b_j - a_j)^2}\right).$$

We start by considering the directed case. Let $C$ be a directed cycle cover with edge weights $w$ such that $(C, w)$ is $\eta_{k,\varepsilon}$-light. We scale the weights such that $w(C) = 1/\eta_{k,\varepsilon}$. Thus, $w(e) \leq 1$ for all $e \in C$.

Let $c_1, \ldots, c_m$ be the cycles of $C$, and consider any cycle $c_j$ of $C$. We choose one edge of $c_j$ for removal uniformly at random. By doing this for $j \in [m]$, we obtain a decomposition $P$ of $C$. Fix any objective $i$. Let $X_j = \sum_{e \in c_j \cap P} w_i(e)$ be the random variable of the contribution of $c_j$ to the weight $w_i(P)$. Since $w_i(e) \in [0, 1]$ for all $e \in C$, there exist $a_j, b_j \in \mathbb{R}$ such that $X_j$ assumes only values in $[a_j, b_j]$ and $0 \leq b_j - a_j \leq 1$. Let $X = \sum_{j=1}^{m} X_j = w_i(P)$ be the random variable of the weight of $P$ with respect to objective $i$. Since every cycle has a length of at least two, the probability of deleting any fixed edge is at most $1/2$. Thus, by linearity of expectation, we have $\mathbb{E}(X) \geq A/2$.

If we can show that $\mathbb{P}(X < (\frac{1}{2} - \varepsilon) \cdot A) < 1/k$, then, by a union bound, $\mathbb{P}(\exists i \in [k] : w_i(P) < (\frac{1}{2} - \varepsilon) \cdot A) < 1$, which would imply the existence of a decomposition $P$ as claimed. Since $0 \leq b_j - a_j \leq 1$, we have

$$w_i(C) = A = \sum_{j=1}^{m} w_i(c_j) \geq \sum_{j=1}^{m} b_j \geq \sum_{j=1}^{m} b_j - a_j \geq \sum_{j=1}^{m} (b_j - a_j)^2.$$

Plugging this into Hoeffding's bound yields

$$\mathbb{P}\left(w_i(P) < \left(\frac{1}{2} - \varepsilon\right) \cdot w_i(C)\right) \leq \exp\left(-\frac{2\varepsilon^2 w_i(C)^2}{w_i(C)}\right) < \frac{1}{k^2}$$

for $k \geq 2$ since $w_i(C) = 2 \ln k/\varepsilon^2 = 1/\eta_{k,\varepsilon}$. The proof for undirected cycle covers is identical except for $\mathbb{E}(X) \geq 2A/3$ and therefore omitted. □

We now know that decompositions exist. But, in order to use them in approximation algorithms, we have to find them efficiently. Theorem 3.1 immediately yields a randomized algorithm: Assume that we have an $\eta_{k,\varepsilon}$-light pair $(C, w)$. We randomly select one edge of every cycle of $C$ for removal and put all remaining edges into $P$. The probability that $P$ is not a $(\frac{1}{2} - \varepsilon)$- or $(\frac{2}{3} - \varepsilon)$-decomposition (depending on whether $C$ is directed or undirected) is bounded from above by $1/k \leq 1/2$. Thus, we obtain a feasible decomposition with constant probability. We iterate this process until a feasible decomposition is found.

For the deterministic algorithm, we assume again that we have an $\eta_{k,\varepsilon}$-light pair $(C, w)$. We scale the weights such that $w_i(C) = 1/\eta_{k,\varepsilon}$ for all $i$. Then $w(e) \leq 1$ for all $e \in C$. The main idea is to reduce an arbitrary instance to a new instance whose size depends only on $k$ and $\varepsilon$.

First, we normalize our cycle cover such that they consist solely of cycles of the shortest possible length. For directed cycle covers $C$, we can restrict ourselves to cycles of length two: Any cycle $c$ of length $\ell$ with edges $e_1, \ldots, e_\ell$ can be replaced by $\lfloor \ell/2 \rfloor$ cycles $(e_{2j-1}, e_{2j})$ for $j = 1, \ldots, \lfloor \ell/2 \rfloor$. If $\ell$ is odd, then we add an edge $e_{\ell+1}$ with $w(e_{\ell+1}) = 0$ and add the cycle $(e_\ell, e_{\ell+1})$. (Technically, edges consist of vertices, and we cannot simply reconnect them. What we mean is that we create new cycles of length two, and the edges of those cycles have the same names and the same weights as in the original cycles.) We do this for all cycles of length at least three and call the resulting cycle cover $C'$. Now any decomposition $P'$ of $C'$ yields a decomposition $P$ of the original cycle cover $C$ by removing the newly added edges $e_{\ell+1}$ if they are in $P'$. Furthermore, $w_i(e) \leq 1$ for the new cycle cover $C'$. Analogously, undirected cycle covers can be normalized to consist solely of cycles of length three.

Second, assume that we have two cycles $c$ and $c'$ in a normalized cycle cover with $w(c) + w(c') \leq 1$. Then we can combine $c$ and $c'$ to $\tilde{c}$: Let $e_1, e_2$ and $e'_1, e'_2$ be the edges of $c$ and $c'$, respectively. Then we can replace $e_i$ and $e'_i$ by $\tilde{e}_i$ with $w(\tilde{e}_i) = w(e_i) + w(e'_i)$. The cycle cover plus edge weights thus obtained are still $\eta_{k,\varepsilon}$-light. We continue combining cycles greedily until no more combinations are possible. For undirected cycles, we proceed analogously. The difference is that the cycles consist of three edges. The resulting cycle cover contains at most $2k/\eta_{k,\varepsilon}$ cycles. Thus, an optimal decomposition can be found with a running-time that now only depends on $k$ and $\varepsilon$.

Overall, for every fixed $\varepsilon > 0$ and $k \geq 2$, we have a deterministic algorithm that, given an $\eta_{k,\varepsilon}$-light directed cycle cover $C$ with edge weights $w$, computes a $(\frac{1}{2} - \varepsilon)$-decomposition of $C$ in time polynomial in the input size. If $C$ is undirected, a $(\frac{2}{3} - \varepsilon)$-decomposition can be computed analogously. We call these algorithms DECOMPOSE with parameters $C$, $w$, and $\varepsilon$.

## 4. APPROXIMATION ALGORITHMS FOR MULTI-CRITERIA MAX-TSP

In this section, MaxCC-Approx denotes the randomized FPTAS for cycle covers. More precisely, let $G$ be a graph (directed or undirected), $w = (w_1, \ldots, w_k)$ be edge weights, $\varepsilon > 0$, and $p > 0$. Then MaxCC-Approx$(G, w, \varepsilon, p)$ yields a $(1-\varepsilon)$-approximate Pareto curve of cycle covers of $G$ with weights $w$ with a success probability of at least $1 - p$.

### 4.1. Multi-Criteria Max-ATSP

Our goal is to guess small sets of heavy edges such that decomposition on the remaining graph is possible. To do so, we need the following notation. For a graph $G = (V, E)$ and a subset $K \subseteq E$ of $G$'s edges ($K$ forms a subset of a Hamiltonian cycle), we get $G_{-K}$ by contracting all edges of $K$. Contracting a single edge $(u, v)$ means removing all outgoing edges of $u$, removing all incoming edges of $v$, and identifying $u$ and $v$. Analogously, for a Hamiltonian cycle $H$ and edges $K$, we obtain a Hamiltonian cycle $H_{-K}$ of $G_{-K}$ by contracting the edges in $K$. If $(G, w)$ is an instance, then $(G_{-K}, w)$ denotes the instance with $w$ modified according to the edge contractions.

We now prove the following: For every Hamiltonian cycle $\tilde{H}$, there exists a (small) set $K \subseteq \tilde{H}$ of edges such that $\tilde{H}_{-K}$ is light-weight. Therefore, let

$$f(k, \varepsilon) = k \cdot \left\lceil \frac{\log(1/2 + \varepsilon)}{\log\bigl(1 - \eta_{k,\varepsilon/2} + (\frac{\varepsilon}{2})^3\bigr)} \right\rceil .$$

From now on, we assume that $\varepsilon \in (0, \frac{1}{2 \ln k})$ is fixed.

Lemma 4.1. *For every Hamiltonian cycle $\tilde{H}$ and every $\varepsilon > 0$, there exists a subset $K \subseteq \tilde{H}$ such that $|K| \leq f(k, \varepsilon)$ and, for every $i \in [k]$, we have at least one of the following two properties:*

*(1)* $w_i(K) \geq (\frac{1}{2} - \varepsilon) \cdot w_i(\tilde{H})$.
*(2)* $w_i(e) \leq \bigl(\eta_{k,\varepsilon/2} - (\frac{\varepsilon}{2})^3\bigr) \cdot w_i(\tilde{H}_{-K})$ *for all* $e \in \tilde{H}_{-K}$.

Proof. We put edges one by one into $K$ until the properties are met for all objectives. If not all $i$ fulfill Property 1 or 2 yet, then we have to add another vertex to $K$. Let $\xi = \eta_{k,\varepsilon/2} - (\frac{\varepsilon}{2})^3$ for short. There exists an edge $e \in \tilde{H} \setminus K$ and an $i \in [k]$ such that $w_i(e) > \xi w_i(\tilde{H}_{-K})$ and $w_i(K) < (\frac{1}{2} - \varepsilon) \cdot w_i(\tilde{H})$. We say that this $i$ is the winner of round $j$, and we add $e$ to $K$. Let us call the new set $K' = K \cup \{e\}$.

Whenever an $i \in [k]$ is a winner, we have

$$\frac{w_i(\tilde{H}_{-K'})}{w_i(\tilde{H}_{-K})} \leq 1 - \xi .$$

By definition, we have $w(K) + w(\tilde{H}_{-K}) = w(\tilde{H})$. Thus, if $i$ has won $\ell$ rounds on the way to $K$, we can conclude that $w_i(K) \geq \bigl(1 - (1 - \xi)^\ell\bigr) \cdot w_i(\tilde{H})$. For $\ell = \lceil \frac{\log 1/2 + \varepsilon}{\log(1-\xi)} \rceil$, we have $w_i(K) \geq (\frac{1}{2} - \varepsilon) \cdot w_i(\tilde{H})$. Observing that every round has a winner completes the proof. □

Now we know that few edges for $K$ suffice to make $\tilde{H}_{-K}$ light-weight. But given the set $K$, how do we find an appropriate cycle cover? Therefore, let $\beta_i = \max\{w_i(e) \mid e \in \tilde{H}_{-K}\}$ be the weight of the heaviest edge with respect to the $i$th objective. Let $\beta = \beta(\tilde{H}_{-K}) = (\beta_1, \ldots, \beta_k)$. We modify our edge weights $w$ to $w^\beta$ as follows:

$$w^\beta(e) = \begin{cases} w(e) & \text{if } w(e) \leq \beta \text{ and} \\ 0 & \text{if } w_i(e) > \beta_i \text{ for some } i. \end{cases}$$

**Algorithm 1** Approximation algorithm for $k$-Max-ATSP.

$\mathcal{P}_{\text{TSP}} \leftarrow$ MAXATSP-APPROX$(G, w, \varepsilon)$
**input:** directed complete graph $G = (V, E)$, $w : E \to \mathbb{Q}_+^k$, $\varepsilon > 0$
**output:** $(\frac{1}{2} - \varepsilon)$ approximate Pareto curve $\mathcal{P}_{\text{TSP}}$ for $k$-Max-ATSP with a probability of at least 1/2

```
for all K ⊆ E with |K| ≤ f(k, ε) that form a subset of a tour and bounds β do
    C_{K,β} ← MaxCC-Approx(G_{−K}, w^β, ε/2, 1/(2n^{2k+f(k,ε)}))
    for all I ⊆ [k] and C ∈ C_{K,β} do
        if w_I^β(e) ≤ η_{k,ε/2} · w_I^β(C) for all e ∈ C then
            P ← Decompose(C, w_I^β, ε/2)
            add edges to P ∪ K to obtain a Hamiltonian cycle H
            add H to P_TSP
```

This means that we set all edge weights exceeding $\beta$ to 0. Since $\tilde{H}_{-K}$ does not contain any of those edges, we have $w(\tilde{H}_{-K}) = w^\beta(\tilde{H}_{-K})$. The advantage of $w^\beta$ is that, if we compute a $(1 - \varepsilon)$ approximate Pareto curve $\mathcal{C}^\beta$ of cycle covers with edge weights $w^\beta$, we obtain a cycle cover to which we can apply decomposition to obtain a collection $P$ of paths. This is stated in the following lemma.

LEMMA 4.2. *Let $\nu > 0$ be arbitrary. Let $H_0$ be a directed Hamiltonian cycle with $w(e) \leq \big(\eta_{k,\nu} - \nu^3\big) \cdot w(H_0)$ for all $e \in H_0$. Let $\beta = \beta(H_0)$, and let $\mathcal{C}$ be a $(1 - \nu)$ approximate Pareto curve of cycle covers with respect to $w^\beta$.*

*Then $\mathcal{C}$ contains a cycle cover $C$ with $w^\beta(C) \geq (1-\nu)\cdot w(H_0)$ and $w^\beta(e) \leq \eta_{k,\nu} \cdot w^\beta(C)$ for all $e \in C$. This cycle cover $C$ yields a decomposition $P \subseteq C$ with $w(P) \geq (\frac{1}{2} - 2\nu) \cdot w(H_0)$.*

PROOF. Since the Hamiltonian cycle $H_0$ is in particular a cycle cover, the set $\mathcal{C}$ contains a cycle cover $C$ with $w^\beta(C) \geq (1 - \nu) \cdot w^\beta(H_0) = (1 - \nu) \cdot w(H_0)$. For every edge $e \in C$ and every $i$, we have $w_i^\beta(e) \leq (\eta_{k,\nu} - \nu^3) \cdot w_i^\beta(H_0) \leq \frac{\eta_{k,\nu} - \nu^3}{1-\nu} \cdot w_i^\beta(C) \leq \eta_{k,\nu} \cdot w_i(C)$. The last inequality follows from $\eta_{k,\nu} - \nu^3 \leq \eta_{k,\nu} \cdot (1 - \nu)$. This is equivalent to $\nu^2 \geq \eta_{k,\nu} = \frac{\nu^2}{2 \ln k}$, which is valid.

The cycle cover $C$ can be decomposed into a collection $P \subseteq C$ of paths with $w(P) \geq w^\beta(P) \geq (\frac{1}{2} - \nu) \cdot w^\beta(C) \geq (\frac{1}{2} - \nu) \cdot (1 - \nu) \cdot w(H_0) \geq (\frac{1}{2} - 2\nu) \cdot w(H_0)$ by Theorem 3.1. □

For $H_0 = \tilde{H}_{-K}$, the set $P$ approximates $\tilde{H}_{-K}$, and $P \cup K$ yields a tour $H$ that approximates $\tilde{H}$. However, to obtain an algorithm, we have to find $\beta$ and $K$. So far, we have assumed that we already know the Hamiltonian cycles that we aim for. But there is only a polynomial number of possibilities for $\beta$ and $K$: For all $\beta$ and for all $i \in [k]$, we can assume that there is an edge with $w_i(e) = \beta_i$. Thus, there are at most $O(n^2)$ choices for $\beta_i$, hence at most $O(n^{2k})$ in total. The cardinality of $K$ is bounded in terms of $f(k, \varepsilon)$ as shown in the lemma above. For fixed $k$ and $\varepsilon$, there is only a polynomial number of subsets of cardinality at most $f(k, \varepsilon)$.

Overall, we obtain MAXATSP-APPROX (Algorithm 1) and the following theorem.

THEOREM 4.3. *For every fixed $k \geq 2$ and $\varepsilon > 0$,* MAXATSP-APPROX *is a randomized $\frac{1}{2} - \varepsilon$ approximation for $k$-criteria Max-ATSP whose running-time is polynomial in the input size.*

PROOF. Let us analyze the approximation ratio first. To do this, we assume that all randomized computations are successful. After that, we analyze success probability and running-time.

Let $\tilde{H}$ be an arbitrary Hamiltonian cycle. We have to show that there exists a Hamiltonian cycle $H \in \mathcal{P}_{\text{TSP}}$ with $w(H) \geq (\frac{1}{2} - \varepsilon) \cdot w(\tilde{H})$. By Lemma 4.1, there exists a set $K \subseteq \tilde{H}$ of cardinality at most $f(k, \varepsilon)$ and a set $I \subseteq [k]$ with the following properties:

— For every $i \in [k] \setminus I$, we have $w_i(K) \geq (\frac{1}{2} - \varepsilon) \cdot w_i(\tilde{H})$.
— For every $i \in I$ and for every edge $e \in \tilde{H}_{-K}$, we have $w_i(e) \leq \bigl(\eta_{k,\varepsilon/2} - (\frac{\varepsilon}{2})^3\bigr) \cdot w_i(\tilde{H}_{-K})$.

Let $\beta = \beta(\tilde{H}_{-K})$. According to Lemma 4.2 with $H_0 = \tilde{H}_{-K}$ and $\nu = \varepsilon/2$, the set $\mathcal{C}_{K,\beta}$ contains a cycle cover $C$ that can be decomposed into a collection $P$ of paths such that $w_i(P) \geq (\frac{1}{2} - \varepsilon) \cdot w_i(\tilde{H}_{-K})$. The set $P \cup K$ is also a collection of paths. We get a Hamiltonian cycle $H \supseteq P \cup K$ by adding arbitrary edges. For the weight of $H$, we have

$$w_i(H) \geq w_i(K) \geq \left(\frac{1}{2} - \varepsilon\right) \cdot w(\tilde{H})$$

for every $i \in I$ and

$$w_i(H) \geq w_i(P) + w_i(K) \geq \left(\frac{1}{2} - \varepsilon\right) \cdot w_i(\tilde{H}_{-K}) + w_i(K) \geq \left(\frac{1}{2} - \varepsilon\right) \cdot w_i(\tilde{H})$$

for every $i \in [k] \setminus I$ since $w_i(K) + w_i(\tilde{H}_{-K}) = w_i(\tilde{H})$. This proves the approximation ratio.

The running time and the error probability remain to be analyzed. The error probabilities of the randomized computations in line 2 are chosen such that they sum up to at most $1/2$. This yields that the probability that one of the computations fails is at most $1/2$. The running time follows since $f(k, \varepsilon)$ is independent of $n$, the number of bounds $\beta$ is bounded by $n^{2k}$, and there are $2^k$ possible sets $I$. □

### 4.2. Multi-Criteria Max-STSP

Our goal is again to show that, for any Hamiltonian cycle $\tilde{H}$, taking out a small set $K$ of heavy edges suffices to make the rest of $\tilde{H}$ light-weight. Unfortunately, contracting heavy edges in undirected graphs is not as easy as it is in directed graphs: The statements "remove all incoming" and "remove all outgoing" edges are not well-defined in an undirected graph.

To circumvent these problems, we do not contract edges $e = \{u, v\}$. Instead, we set the weight of all edges incident to $u$ or $v$ to 0. This allows us to add the edge $e$ to any collection $P$ of paths without decreasing the weight: We remove all edges incident to $u$ or $v$ from $P$, and then we add $e$. The result is again a collection of paths.

However, by setting the weight of edges adjacent to $u$ or $v$ to 0, we might destroy a lot of weight with respect to some objective. To circumvent this problem as well, we put larger neighborhoods of the edges into $K$. In this way, we can add our heavy-weight edge (plus some more edges of its neighborhood) to the collection of paths without losing too much weight from removing other edges. Lemma 4.4 below justifies this.

LEMMA 4.4. *Let $\tilde{H}$ be a Hamiltonian cycle as described above, let $w = (w_1, \ldots, w_k)$ be edge weights, and let $e_1, \ldots, e_\ell$ be any $\ell$ distinct edges of $\tilde{H}$. Then there exists a $j \in [\ell]$ such that*

$$w(e_j) \leq \frac{k}{\ell} \cdot w(\tilde{H}).$$

PROOF. Suppose otherwise and assume without loss of generality that $w_i(\tilde{H}) > 0$ for all $i$. We scale the weights such that $w_i(\tilde{H}) = 1$ for all $i$. Then for all $j$ there is an $i_j$ with $w_{i_j}(e_j) > \frac{k}{\ell} \cdot w_{i_j}(\tilde{H}) = \frac{k}{\ell}$. Thus, $\sum_{j=1}^{\ell} \sum_{i=1}^{k} w_i(e_j) > \sum_{j=1}^{\ell} \frac{k}{\ell} \cdot w_{i_j}(\tilde{H}) = k$. But, since all edges are distinct, we also have $\sum_{j=1}^{\ell} \sum_{i=1}^{k} w_i(e_j) \leq \sum_{i=1}^{k} w_i(\tilde{H}) = k$ – a contradiction. □

Let $\tilde{H}$ be an arbitrary Hamiltonian cycle. Let $e_0, e_1, \ldots, e_{n-1}$ be the edges of $\tilde{H}$ in the order in which they appear in $\tilde{H}$ ($e_0$ is chosen arbitrarily). Let $e_j = \{v_j, v_{j+1}\}$, where arithmetic of the indices here and in the following is modulo $n$. Now let $e_0$ be a heavy-weight edge of $\tilde{H}$. Then we put $e_0$ into our set $K$, and we set the weight of all edges incident to $v_0$ and $v_1$ to 0. But in this way, we lose the weight of $e_1$ and $e_{-1}$. In order to maintain the approximation ratio, we have to avoid that we lose too much weight. Therefore, we consider paths that include $e_0$. If we set the weight of all edges incident to $v_{p+1}, \ldots, v_q$ with $p < 0 < q$ to 0, we lose only the weight of $e_p$ and $e_q$. To keep track of things, we also put $e_{p+1}, \ldots, e_{q-1}$ into $K$. Furthermore, we put the two edges $e_p$ and $e_q$, whose weight might get lost, into a set $T$. By Lemma 4.4, we can make sure that both $e_p$ and $e_q$ are not too heavy. Finally, we put $v_{p+1}, \ldots, v_q$ into the set $L$, which is the set of vertices whose adjacent edges have now weight 0. We denote the corresponding edge weights by $w^L$ (see below for a formal definition).

Given any collection of paths $P$, we can now remove all edges of weight 0 and add the edges $e_{p+1}, \ldots, e_{q-1}$ to obtain a new collection of paths. This does not change the weight with respect to $w^L$. The only edges that we cannot force to be in $H$ are $e_p$ and $e_q$. In order to maintain a good approximation ratio we have to make sure that both are light with respect to all objectives. This is where Lemma 4.4 comes into play: If we choose $p$ and $q$ sufficiently large, then $e_p$ and $e_q$ are light.

To fix notation, let $K$ be the of edges that we want to keep of $\tilde{H}$. Let $L = L(K) = \{v \in V \mid \exists e \in K : v \in e\}$ be the set of vertices incident to edges in $K$. Let $w^{-L}$ be defined by

$$w^{-L}(e) = \begin{cases} w(e) & \text{if } e \cap L = \emptyset \text{ and} \\ 0 & \text{if } e \cap L \neq \emptyset. \end{cases}$$

This means that the weight of edges incident to $L$ is set to 0, which includes the edges in $K$. For a bound $\beta$, $w^{-L,\beta}(e)$ is defined accordingly: $w^{-L,\beta}(e) = w^{-L}(e)$ if $w^{-L}(e) \leq \beta$ and $w^{-L,\beta}(e) = 0$ otherwise.

There are more edges of $\tilde{H} \setminus K$ whose weight is affected by $w^{-L}$: Let

$$T = T(K) = \{e \in \tilde{H} \mid e \notin K, e \cap L(K) \neq \emptyset\}$$

be the set of edges that have exactly one endpoint in $L$. The weights of these edges are set to 0 in $w^{-L}$, but we cannot force them to be in any cycle cover as mentioned above. (They are the edges of $\tilde{H}$ that are adjacent to $K$ but not in $K$.)

The following lemma is the undirected counterpart of Lemma 4.1. In particular, it takes care of the set $T$, which is only needed for the analysis and not for the algorithm. Let

$$g(k, \varepsilon) = k \cdot \left\lceil \frac{\log(1/3)}{\log\bigl(1 - \eta_{k,\varepsilon/3} + (\frac{\varepsilon}{3})^3\bigr)} \right\rceil .$$

The function $g$ plays the same role as the function $f$ in the previous subsection.

LEMMA 4.5. *For every Hamiltonian cycle $\tilde{H}$ and every $\varepsilon > 0$, there exists a subset $K \subseteq \tilde{H}$ of at most $g(k, \varepsilon)$ paths, each of length at most $\frac{12k}{\varepsilon} g(k, \varepsilon)$ with the following properties: Let $L = L(K)$ and $T = T(K)$. For every $i \in [k]$, we have at least one of the following two properties:*

*(1)* $w_i(K) \geq (2/3 - \varepsilon) \cdot w_i(\tilde{H})$.
*(2)* $w_i^{-L}(e) \leq \bigl(\eta_{k,\varepsilon/3} - (\frac{\varepsilon}{3})^3\bigr) \cdot w_i^{-L}(\tilde{H})$ *for all* $e \in \tilde{H}$.

*Furthermore, we have* $w(T) \leq \frac{\varepsilon}{3} \cdot w(\tilde{H})$.

PROOF. The proof is similar to the proof of Lemma 4.1, but slightly more involved since we have to keep track of the set $T$. Let $\xi = \eta_{k,\varepsilon/3} - (\frac{\varepsilon}{3})^3$.

We put paths one by one into $K$ until the properties are met. If not all $i$ fulfill Property 1 or 2 yet, then we have to add another path to $K$. In this case, there exists an edge $e_0 \in \tilde{H}_{-K}$ and an $i \in [k]$ such that $w_i(e_0) > \xi \cdot w_i^{-K}(\tilde{H})$ and $w_i(K) < (\frac{2}{3} - \varepsilon) \cdot w_i(\tilde{H})$. We call $i$ the winner of round $j$ and add it to $K$. Furthermore, we extend $e_0$ to both sides to obtain paths $e_p, e_{p+1}, \ldots, e_0, \ldots, e_q$ for some $p < 0 < q$. Here, $p$ is chosen such that either $w(e_p) \leq \frac{\varepsilon}{6g(k,\varepsilon)} \cdot w(\tilde{H})$ or $e_p \in K \cup T$, and $q$ is chosen analogously. We can choose $-p, q \leq \frac{6kg(k,\varepsilon)}{\varepsilon}$. We put $e_p$ and $e_q$ into $T$ and $e_{p+1}, \ldots, e_{q-1}$ into $K$.

Now let $K$ be the set of edges before an iteration, and let $K' = K \cup \{e_{p+1}, \ldots, e_{q-1}\}$ be the set of edges afterwards. We have $w_i^{-K'}(\tilde{H}) + w_i(e_0) \leq w_i^{-K}(\tilde{H})$. Since $w_i(e_0) > \xi w_i^{-K}$, this yields $w_i^{-K'}(\tilde{H}) < (1 - \xi) \cdot w_i^{-K}(\tilde{H})$. Thus, if $i$ is the winner in $\ell$ rounds, and the resulting set of edges is $K$, then $w_i^{-K}(\tilde{H}) \leq (1-\xi)^\ell \cdot w_i(\tilde{H})$. If $\ell = \lceil \log_{1-\xi} 1/3 \rceil = g(k,\varepsilon)/k$, then

$$w_i^{-K}(\tilde{H}) \leq \frac{1}{3} \cdot w_i(\tilde{H}) \,. \tag{1}$$

Since every round has a winner, after at most $g(k,\varepsilon)$, all properties are met. This is because $w(\tilde{H}) = w(K) + w(T) + w^{-K}(\tilde{H})$. Any edge $e \in T$ fulfills $w(e) \leq \frac{\varepsilon}{6g(k,\varepsilon)} \cdot w(\tilde{H})$. Since we put at most $2g(k,\varepsilon)$ edges into $T$, we have $w(T) \leq \frac{\varepsilon}{3} \cdot w(\tilde{H})$. Together with (1), this concludes the proof. □

Again, given that we have the correct set $K \subseteq \tilde{H}$, we have to find a cycle cover that approximates $\tilde{H}$ with respect to $w^{-L(K)}$. That this can be done is shown in the following undirected counterpart of Lemma 4.2.

LEMMA 4.6. *Let $\nu > 0$ be arbitrary. Let $\tilde{H}$ be an undirected Hamiltonian cycle with $w(e) \leq \big(\eta_{k,\nu} - \nu^3\big) \cdot w(\tilde{H})$ for all $e \in \tilde{H}$. Let $\beta = \beta(\tilde{H})$, and let $\mathcal{C}$ be a $(1-\nu)$ approximate Pareto curve of cycle covers with respect to $w^\beta$.*

*Then $\mathcal{C}$ contains a cycle cover $C$ with $w^\beta(C) \geq (1-\nu) \cdot w(\tilde{H})$ and $w^\beta(e) \leq \eta_{k,\nu} \cdot w^\beta(C)$ for all $e \in C$. This cycle cover $C$ yields a decomposition $P \subseteq C$ with $w(P) \geq (\frac{2}{3} - 2\nu) \cdot w(\tilde{H})$.*

PROOF. The proof is almost identical to the proof of Lemma 4.2 and thus omitted. □

Now we have everything for algorithm MAXSTSP-APPROX (Algorithm 2) and Theorem 4.7.

THEOREM 4.7. *For every $k \geq 2$ and $\varepsilon > 0$, MAXSTSP-APPROX is a randomized $\frac{2}{3} - \varepsilon$ approximation for k-criteria Max-STSP whose running-time is polynomial in the input size.*

PROOF. Let us first concentrate on the approximation ratio. Consider any Hamiltonian cycle $\tilde{H}$. We have to show that $\mathcal{P}_{\text{TSP}}$ contains a Hamiltonian cycle $H$ with $w(H) \geq (2/3 - \varepsilon)w(\tilde{H})$. According to Lemma 4.5, there exists a set $K \subseteq \tilde{H}$ and a subset $I \subseteq [k]$ of the objectives such that the following holds:

— For every $i \in [k] \setminus I$, we have $w_i(K) \geq (2/3 - \varepsilon) \cdot w_i(\tilde{H})$.
— For every $i \in I$ and for every edge $e \in H_{-K}$, we have $w_i(e) \leq \big(\eta_{k,\varepsilon/3} - (\frac{\varepsilon}{3})^3\big) \cdot w_i(H_{-K})$.

Let $L = L(K)$, and let $\beta = \beta(\tilde{H})$ with respect to the edge weights $w^{-L}$. According to Lemma 4.6 with edge weights $w^{-L}$ and $\nu = \varepsilon/3$, the set $\mathcal{C}_{L,\beta}$ contains a cycle cover $C$ from which we get a decomposition $P \subseteq C$ with $w_i^{-L}(P') = w_i^{-L}(P) \geq (\frac{2}{3} - \frac{2\varepsilon}{3}) \cdot w_i^{-L}(\tilde{H})$. ($P'$ is obtained from $P$ by removing edges of weight 0.) Since $w(\tilde{H}) = w^{-L}(\tilde{H}) + w(T) + w(K)$ and

**Algorithm 2** Approximation algorithm for $k$-Max-STSP.

$\mathcal{P}_{\text{TSP}} \leftarrow$ MaxSTSP-Approx$(G, w, \varepsilon)$
**input:** undirected complete graph $G = (V, E)$, $w : E \to \mathbb{Q}_+^k$, $\varepsilon > 0$
**output:** $(\frac{2}{3} - \varepsilon)$ approximate Pareto curve $\mathcal{P}_{\text{TSP}}$ for $k$-Max-STSP with a probability of at least 1/2

```
1: for all K ⊆ E that consist of at most g(k, ε) paths of length at most ≤ 12kg(k, ε)/ε do
2:     for all bounds β do
3:         L ← L(K)
4:         C_{L,β} ← MaxCC-Approx(G, w^{−L,β}, ε/3, 1/(2n^{2k+12kg²(k,ε)/ε}))
5:         for all I ⊆ [k] and C ∈ C_{L,β} do
6:             if w_I^{−L,β}(e) ≤ η_{k,ε/3} · w_I^{−L,β}(C) for all e ∈ C then
7:                 P ← Decompose(C, w_I^{−L,β}, ε/4)
8:                 remove edges of weight 0 from P to get P′
9:                 add edges to K ∪ P′ to get a Hamiltonian cycle H; add H to P_TSP
```

$w(T) \leq \frac{\varepsilon}{3} \cdot w(\tilde{H})$ according to Lemma 4.5, we have $w(H) \geq w(P' \cup K) \geq (2/3 - \varepsilon) \cdot w(\tilde{H})$, which is enough.

The error probabilities of the randomized computations in line 4 sum up to at most 1/2 since there are at most $n^{2k}$ bounds $\beta$ and at most $n^{6kg^2(k,\varepsilon)}$ subsets $K$. By a union bound, the probability that one of the computations fails is thus at most 1/2. The running time follows since $g(k, \varepsilon)$ is independent of $n$, the number of bounds $\beta$ is bounded by $n^{2k}$, and the number of $I$ is $2^k$. $\square$

## 5. DETERMINISTIC APPROXIMATIONS FOR 2-MAX-STSP

The algorithms presented in the previous section are randomized due to the computation of approximate Pareto curves of cycle covers. So are most approximation algorithms for multi-criteria TSP with the exception of a simple $(2 + \varepsilon)$ approximation for $k$-Min-STSP [Manthey and Ram 2009].

As a first step towards deterministic approximation algorithms for multi-criteria Max-TSP, we present a deterministic $7/27 \approx 0.26$ approximation for 2-Max-STSP. The key insight for the results of this section is the following lemma.

LEMMA 5.1. *Let $M$ be a (not necessarily perfect) matching, let $H$ be a collection of paths or a Hamiltonian cycle, and let $w$ be edge weights ($w$ is a single-criterion function). Then there exists a subset $P \subseteq H$ such that*

*(1) $P \cup M$ is a collection of paths or a Hamiltonian cycle (we call $P$ in this case an $M$-feasible set) and*
*(2) $w(P) \geq w(H)/3$.*

PROOF. We prove the lemma by induction on $|M| + |H|$. For $|M| + |H| = 0$, the lemma is trivially true. As induction hypothesis, we assume the lemma holds for all $M$ and $H$ with $|M| + |H| < \ell$, and we want to prove it for $|M| + |H| = \ell$.

We distinguish two cases. The first case is $M \cap H \neq \emptyset$. Then we set $\tilde{P} = M \cap H$ and $H' = H \setminus M$. By induction hypothesis, there exists an $M$-feasible $P' \subseteq H'$ such that $w(P') \geq w(H')/3$. Since $\tilde{P} \subseteq M$, also $P = P' \cup \tilde{P}$ is $M$-feasible. Observing that $w(P) = w(P') + w(\tilde{P}) \geq w(H \cap M) + w(H \setminus M)/3 \geq w(H)/3$ completes this case.

The second case is that $M$ and $H$ are disjoint. Let $e = \text{argmax}\{w(e) \mid e \in H\}$ be a heaviest edge of $H$, and let $f_1, f_2 \in H$ be the two edges of $H$ that are incident to $e$. Let $H' = H \setminus \{e, f_1, f_2\}$. (It can happen that $f_1$ or $f_2$ do not exist, namely if $H$ is not a Hamiltonian cycle but a collection of paths. But this is fine.)

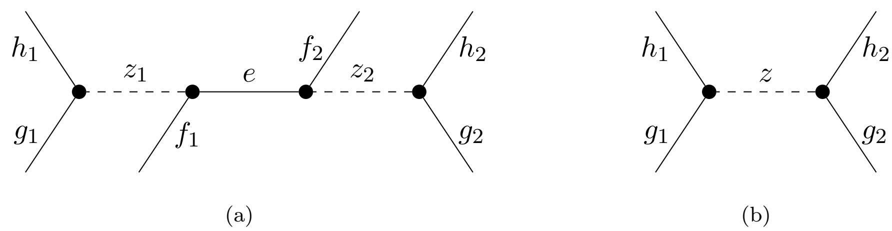


Fig. 1. Contraction for the proof of Lemma 5.1: We keep $e$ and remove $f_1, f_2$ (1(a)). Then we can contract $z_1$ and $z_2$ to $z$ (1(b)).

Let us first treat the case that $e$ is incident to two edges $z_1, z_2 \in M$ of the matching. Then we contract $z_1$ and $z_2$ to a single edge $z$ that connects the two endpoints of $z_1$ and $z_2$ that are not incident to $e$ and remove the two vertices incident to $e$ (see Figure 1). Let $M' = (M \setminus \{z_1, z_1\}) \cup \{z\}$. Since $e, f_1, f_2$ are removed, $H'$ and $M'$ are a valid instance for the lemma, i.e., $M'$ is a matching and $H'$ is a collection of paths ($H'$ cannot be a Hamiltonian cycle). We can apply the induction hypothesis since $|M'| + |H'| < \ell$.

In this way, we obtain an $M'$-feasible set $P' \subseteq H'$ with $w(P') \geq w(H')/3$. Set $P = P' \cup \{e\}$. Since $w(e) \geq w(\{e, f_1, f_2\})/3$, we have $w(P) \geq w(H)/3$. Since $P'$ is $M'$-feasible, the set $P$ is $M$-feasible by construction.

What remains to be considered is the case the $e$ is not incident to two edges $z_1, z_2 \in M$. Then we consider the shortest path in $e_1, \ldots, e_q \in H$ of edges in $H$ that includes $e$ such that $e_1$ and $e_q$ are incident to any edges $z_1, z_2 \in M$. The reasoning above holds in exactly the same way if replace $e$ by the path $e_1, \ldots, e_q$, and we put $e_1, \ldots, e_q$ into $P$. If no such path exists, then either $M = \emptyset$, which is easy to handle, or the path containing $e$ ends somewhere at a vertex of degree 1 in $H \cup M$. In the latter case, we can simply put the whole path into $P$. □

Lemma 5.1 yields tight bounds for the existence of approximate Pareto curves with only a single element. This is the purpose of the following theorem.

THEOREM 5.2.

(1) *For every undirected complete graph $G$ with edge weights $w_1$ and $w_2$, there exists a Hamiltonian cycle $H$ such that $\{H\}$ is a $1/3$ approximate Pareto curve for 2-Max-STSP.*
(2) *Part (1) is tight: There exists a graph $G$ with edge weights $w_1$ and $w_2$ such that, for all $\varepsilon > 0$, no single Hamiltonian tour of $G$ is a $(1/3 + \varepsilon)$ approximate Pareto curve.*

Before embarking on the proof of the theorem, let us remark that single-element approximate Pareto curves exist for no other variant of multi-criteria TSP than 2-Max-STSP: For $k$-Max-STSP for $k \geq 3$, we can consider a vertex incident to three edges of weight $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, respectively. All other edges of the graph have weight 0. Then no single Hamiltonian cycle can have positive weight with respect to all three objectives simultaneously. Similarly, no such result is possible for $k$-Max-ATSP and for $k$-Min-TSP for any $k \geq 2$.

PROOF. Let $H_1$ and $H_2$ be Hamiltonian cycles of $G$ such that $H_1$ maximizes $w_1$ and $H_2$ maximizes $w_2$. Then there exists a matching $M \subseteq H_1$ with $w(M) \geq w(H_1)/3$. (We can actually get $w(H_1)/2$ if $G$ has an even number of vertices and $\frac{n-1}{2n} \cdot w(H_1)$ if the number $n$ of $G$'s vertices is odd. This, however, does not improve the result.) We apply Lemma 5.1 with $H = H_2$ and obtain an $M$-feasible set $P \subseteq H_2$. From $M$ and $P$, we obtain a Hamiltonian cycle $H' \supseteq M \cup P$: Either $M \cup P$ is already a Hamiltonian cycle, then nothing has to be

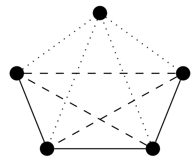

Fig. 2. The graph for Theorem 5.2(2). Solid edges are of weight $(1, 0)$, dashed edges are of weight $(0, 1)$, dotted edges have weight $(0, 0)$.

**Algorithm 3** Approximation algorithm for 2-Max-STSP.

$\mathcal{P}_{\text{TSP}} \leftarrow$ BIMAXSTSP-APPROX$(G, w_1, w_2)$
**input:** undirected complete graph $G = (V, E)$, edge weights $w_1, w_2 : E \to \mathbb{Q}_+^k$
**output:** a 7/27 approximate Pareto curve $H$ for 2-Max-STSP
compute a maximum-weight matching $M$ with respect to $w_1$
compute a 7/9 approximate Hamiltonian cycle $H_2$ with respect to $w_2$
$P \leftarrow H_2 \cap M$
$M' \leftarrow M$
$H_2 \leftarrow H_2 \setminus P$
**while** $H_2 \neq \emptyset$ **do**
  $e \leftarrow \text{argmax}\{w_2(e') \mid e' \in H_2\}$
  extend $e$ to a path $e_1, \ldots, e_q \in H_2$ such that only $e_1$ and $e_q$ are incident to edges $z_1, z_2 \in M'$ or the path cannot be extended anymore
  $P \leftarrow P \cup \{e_1, \ldots, e_q\}$
  $H_2 \leftarrow H_2 \setminus \{e_1, \ldots, e_q\}$
  **if** $z_1$ or $z_2$ exists **then**
    let $f_1, f_2 \in H_2$ be the two edges extending the path if they exist
    $H_2 \leftarrow H_2 \setminus \{f_1, f_2\}$
    **if** both $z_1$ and $z_2$ exist **then**
      contract $z_1$ and $z_2$ to $z$
      $M' \leftarrow (M' \setminus \{z_1, z_2\}) \cup \{z\}$
let $H$ be a Hamiltonian cycle obtained from $P \cup M$

done. Or $M \cup P$ is a collection of paths. Then we add appropriate edges to obtain $H'$. We claim that $\{H'\}$ is a 1/3 approximate Pareto curve: Let $\tilde{H}$ be any Hamiltonian tour. Then

$$w_1(H') \geq w_1(M) \geq w_1(H_1)/3 \geq w_1(\tilde{H})/3$$

and

$$w_2(H') \geq w_2(P) \geq w_2(H_2)/3 \geq w_2(\tilde{H})/3.$$

To finish the proof, let us show that ratio 1/3 is tight. Consider the graph in Figure 2. The solid edges plus two dotted edges form a Hamiltonian cycle of weight $(3, 0)$. The dashed edges plus two other dotted edges form a Hamiltonian cycle of weight $(0, 3)$. To get a single-element $(\frac{1}{3} + \varepsilon)$-approximate Pareto curve $\{H\}$, we must have $w_i(H) \geq 1 + 3\varepsilon$ for $i \in \{1, 2\}$. Thus, the Hamiltonian cycle $H$ must contain two solid edges and two dashed edges, which is impossible. □

Lemma 5.1 and Theorem 5.2 are constructive in the sense that, given a Hamiltonian cycle $H_2$ that maximizes $w_2$, the tour $H$ can be computed in polynomial time. A matching $M$ with $w_1(M) \geq w_1(H_1)/3$ can be computed in cubic time. However, since we cannot compute an optimal $H_2$ efficiently, the results cannot be exploited directly to get an algorithm. Instead, we use an approximation algorithm for finding a Hamiltonian tour with as much weight with respect to $w_2$ as possible. Using the 7/9 approximation algorithm for Max-STSP [Paluch

**Algorithm 4** Approximation algorithm for $k$-Min-ATSP.

$\mathcal{P}_{\text{TSP}} \leftarrow$ MinATSP-Approx$(G, w, \varepsilon)$
**input:** directed complete graph $G = (V, E)$, $n = |V|$, edge weights $w : E \to \mathbb{Q}_+^k$, $\varepsilon > 0$
**output:** $(\log n + \varepsilon)$ approximate Pareto curve for $k$-Min-ATSP with a probability of at least $1/2$

$\varepsilon' \leftarrow \varepsilon^2 / \log^3 n$
$\mathcal{P}_0 \leftarrow \{\emptyset\}$
**for** $j \leftarrow 1$ to $\lfloor \log_2 n \rfloor$ **do**
  $\mathcal{P}_j \leftarrow \emptyset$
  **for all** $C \in \mathcal{P}_{j-1}$ **do**
    **if** $(V, C)$ is connected **then**
      add $C$ to $\mathcal{P}_j$
    **else**
      select one vertex of every component of $(V, C)$ to obtain $V'$
      $\mathcal{C} \leftarrow$ MinCC-Approx$(V', w, \varepsilon', \frac{1}{2Q \log n})$  ▷ $Q$ is defined in Lemma 6.2
      $\mathcal{P}_j \leftarrow \mathcal{P}_j \cup \{C \cup C' \mid C' \in \mathcal{C}\}$
  **while** there are $C, C' \in \mathcal{P}_j$ with the same $\varepsilon'$-signature **do**
    remove one of them arbitrarily
  $j \leftarrow j + 1$
$\mathcal{P}_{\text{TSP}} \leftarrow \emptyset$
**for all** $C \in \mathcal{P}_{\lfloor \log_2 n \rfloor}$ **do**
  walk along the Eulerian cycle of $C$, take shortcuts to obtain a Hamiltonian cycle $H$
  add $H$ to $\mathcal{P}_{\text{TSP}}$

et al. 2009], we obtain Algorithm 3 (which is an algorithmic version of Lemma 5.1) and the following theorem.

THEOREM 5.3. BiMaxSTSP-Approx *is a deterministic* $7/27$ *approximation algorithm with running-time* $O(n^3)$ *for* 2*-Max-STSP.*

PROOF. The running-time is dominated by the running-time of the $7/9$ approximation for Max-STSP by Paluch et al. [2009] and the time for computing the matching, both of which is $O(n^3)$. The approximation ratio follows from $\frac{7}{9} \cdot \frac{1}{3} = \frac{7}{27}$. □

## 6. APPROXIMATION ALGORITHM FOR MULTI-CRITERIA MIN-ATSP

Now we turn to $k$-Min-ATSP, i.e., Hamiltonian cycles of *minimum* weight are sought in directed graphs. Algorithm 4 is an adaptation of the algorithm of Frieze et al. [1982] to multi-criteria ATSP. Therefore, we briefly describe their algorithm: We compute a cycle cover of minimum weight. If this cycle cover is already a Hamiltonian cycle, then we are done. Otherwise, we choose an arbitrary vertex from every cycle. Then we proceed recursively on the subset of vertices thus chosen to obtain a Hamiltonian cycle that contains all these vertices. The cycle cover plus this Hamiltonian cycle form an Eulerian graph. We traverse the Eulerian cycle and take shortcuts whenever visiting vertices more than once. The approximation ratio achieved by this algorithm is $\log_2 n$ for Min-ATSP [Frieze et al. 1982].

To approximate $k$-Min-ATSP, we use MinATSP-Approx (Algorithm 4), which proceeds as follows: We compute an approximate Pareto curve of cycle covers of minimum weight. This is done by MinCC-Approx, where MinCC-Approx$(G, w, \varepsilon, p)$ computes a $(1 + \varepsilon)$ approximate Pareto curve of cycle covers of $G$ with weights $w$ with a success probability of at least $1 - p$ in time polynomial in the input size, $1/\varepsilon$, and $\log(1/p)$. Then we iterate by computing approximate Pareto curves of cycle covers on vertex sets $V'$ for every cycle cover $C$ in the previous set. The set $V'$ contains exactly one vertex of every cycle of $C$. Unfortunately, it can happen that we construct a super-polynomial number of solutions in this way. To cope with this, we remove some intermediate solutions if there are other intermediate solutions whose weight is close by. We call this process *sparsification*. It is performed in lines 12 and 13 of Algorithm 4 and based on the following observation: Let $\varepsilon > 0$, and consider a set $H$ of edges of weight $w(H) \in \mathbb{Q}_+^k$. For every $i \in \{1, \ldots, k\}$ with $w_i(H) \neq 0$, there is a unique $\ell_i \in \mathbb{N}$ such that $w_i(H) \in \big[(1+\varepsilon)^{\ell_i}, (1+\varepsilon)^{\ell_i+1}\big)$. If $w_i(H) = 0$, then we let $\ell_i = -\infty$. We call the vector $\ell = (\ell_1, \ldots, \ell_k)$ the *$\varepsilon$-signature* of $H$ and of $w(H)$. Since $w(H) \in \big([2^{-p(N)}, 2^{p(N)}] \cup \{0\}\big)^k$, where $N$ is the size of the instance, the number of possible values of $\ell_i$ is bounded by a polynomial $q(N, 1/\varepsilon)$. There are at most $q^k$ different $\varepsilon$-signatures, which is polynomial for fixed $k$. To get an approximate Pareto curve, we can restrict ourselves to at most one solution for any $\varepsilon$-signature.

The set $\mathcal{P}_0$ is initialized with the empty set of edges. In the loop in lines 3 to 14, the algorithm computes iteratively Pareto curves of cycle covers. The set $\mathcal{P}_j$ contains sets $C$ of edges consisting of cycle covers: Given a $C \in \mathcal{P}_{j-1}$, $\mathcal{P}_j$ contains cycle covers on the graph consisting of one node for every connected component of $C$. If $(V, C)$ is already connected, then $C$ is simply put into $\mathcal{P}_j$ without modification. In lines 12 and 13, the sparsification takes place. Finally, in lines 15 to 18, Hamiltonian cycles are constructed from the Eulerian graphs.

Let us now come to the analysis of the algorithm. Our goal is to prove the following result, which follows from Lemmas 6.2, 6.3, and 6.5 below. MinATSP-Approx is the first approximation algorithm for $k$-Min-ATSP.

THEOREM 6.1. *For every $\varepsilon > 0$ and $k \geq 2$, Algorithm 4 is a randomized $(\log n + \varepsilon)$ approximation for $k$-Min-ATSP with a success probability of at least $1/2$. Its running-time is polynomial in the input size and $1/\varepsilon$.*

We observe that for every $j \in \{0, 1, \ldots, \lfloor \log_2 n \rfloor\}$ and $C \in \mathcal{P}_j$, the graph $(V, C)$ consists of at most $n/2^j$ connected components. For $j = 0$, this holds since $(V, \emptyset)$ consists of $n$ connected components. For $j > 0$ and $C \in \mathcal{P}_{j-1}$, $(V, C)$ consists of at most $n/2^{j-1}$ connected components by the induction hypothesis. If $(V, C)$ is connected, then $C \in \mathcal{P}_j$, and the claim holds since $n/2^j \geq n/2^{\lfloor \log_2 n \rfloor} \geq 1$. Otherwise, since every cycle involves at least two vertices, the claim holds also for $\mathcal{P}_j$. This yields that $(V, C)$ is connected for all $C \in \mathcal{P}_{\lfloor \log_2 n \rfloor}$: Such a $(V, C)$ consists of at most $n/2^j \leq n/2^{\lfloor \log_2 n \rfloor} < 2$ connected components.

Let us now analyze the running-time. After that, we examine the approximation performance and finally the success probability.

LEMMA 6.2. *The running-time of Algorithm 4 is polynomial in the input size and $1/\varepsilon$.*

PROOF. Let $N$ be the size of the instance at hand, and let $Q = Q(N, 1/\varepsilon')$ be a two-variable polynomial that bounds the number of different $\varepsilon'$-signatures of solutions for instances of size at most $N$. We abbreviate "polynomial in the input size and $1/\varepsilon$" simply by "polynomial." This is equivalent to "polynomial in the input size and $1/\varepsilon'$" by the choice of $\varepsilon'$.

The approximate Pareto curves can be computed in polynomial time with a success probability of at least $1 - (2Q \log n)^{-1}$ by executing the randomized FPTAS $\lceil \log(2Q \log n) \rceil$ times. Thus, all operations can be implemented to run in polynomial time provided that the cardinalities of all sets $\mathcal{P}_j$ are bounded from above by a polynomial $Q$ for all $j$. Then, for each $j$, at most $Q$ approximate Pareto curves of cycle covers are constructed in line 10.

For every $\varepsilon'$-signature and every $j$, the set $\mathcal{P}_j$ contains at most one set of edges for any $\varepsilon'$-signature. The lemma follows since the number of different $\varepsilon'$-signatures is bounded by $Q$. □

Let us now analyze the approximation ratio. To do so, we will assume that all randomized computations of $(1+\varepsilon')$ approximate cycle covers are successful.

LEMMA 6.3. *Assume that in all executions of line 10 of Algorithm 4 an $(1+\varepsilon')$ approximate Pareto curve of cycle covers is successfully computed. Then Algorithm 4 achieves an approximation ratio of $\log_2 n + \varepsilon$ for k-Min-ATSP.*

PROOF. Let $\tilde{H}$ be any Hamiltonian cycle on $V$. We have to show that the set $\mathcal{P}_{\mathrm{TSP}}$ of solutions computed by Algorithm 4 contains a Hamiltonian cycle $H$ with $w(H) \leq (\log_2 n + \varepsilon) \cdot w(\tilde{H})$.

Given any $C \in \mathcal{C}_{\lfloor \log_2 n \rfloor}$, due to the triangle inequality, we construct a Hamiltonian cycle $H$ in lines 15 to 18 such that $w(H) \leq w(C)$. What remains to be proved is that, for every Hamiltonian cycle $\tilde{H}$, there exists a $C \in \mathcal{C}_{\lfloor \log_2 n \rfloor}$ such that $w(C) \leq (\log n + \varepsilon) \cdot w(\tilde{H})$.

LEMMA 6.4. *For every $j$, there exists a $C \in \mathcal{P}_j$ with $w(C) \leq (1+\varepsilon')^j \cdot j \cdot w(\tilde{H})$.*

PROOF. The proof is by induction on $j$. For $j = 0$, the lemma holds since $w(\emptyset) = 0$.

Now assume that the lemma holds for $j-1$ for $j > 0$. Consider any $C \in \mathcal{P}_{j-1}$ that satisfies the lemma for $j-1$ and $\tilde{H}$. Such a $C$ exists by the induction hypothesis. If $(V, C)$ is connected, then $C \in \mathcal{P}_j$, and $C$ satisfies the lemma also for $j$. Otherwise, let $V'$ be the set of vertices chosen from the connected components of $(V, C)$ in line 9. Let $\tilde{H}'$ be $\tilde{H}$ restricted to $V'$ by taking shortcuts. By the triangle inequality, we have $w(\tilde{H}') \leq w(\tilde{H})$. After line 10, $\mathcal{C}$ contains a cycle cover $C'$ with $w(C') \leq (1+\varepsilon') \cdot w(\tilde{H}')$. Then

$$w(C' \cup C) \leq \big((1+\varepsilon')^{j-1} \cdot (j-1) + (1+\varepsilon')\big) \cdot w(\tilde{H}).$$

What remains to be analyzed is the sparsification in lines 12 to 13. After that $\mathcal{P}_j$ contains a $C''$ (with might coincide with $C \cup C'$) with with the same $\varepsilon'$-signature as $C \cup C'$. Thus,

$$\begin{aligned} w(C'') &\leq (1+\varepsilon') \cdot w(C \cup C'') \leq (1+\varepsilon') \cdot \big((1+\varepsilon')^j \cdot j + (1+\varepsilon')\big) \cdot w(\tilde{H}) \\ &\leq (1+\varepsilon')^{j+1} \cdot (j+1) \cdot w(\tilde{H}), \end{aligned}$$

and $C''$ fulfills the requirements of the lemma. □

Since every $C \in \mathcal{C}_{\lfloor \log_2 n \rfloor}$ yields a Hamiltonian cycle without increasing the weight, we obtain an approximation ratio of

$$\log_2 n \cdot (1+\varepsilon')^{\lfloor \log n \rfloor} \leq \log_2 n \cdot \left(1 + \frac{\varepsilon^2}{\log^3 n}\right)^{\log n} \leq \log_2 n \cdot \exp\left(\frac{\varepsilon^2}{\log^2 n}\right) \leq \log_2 n + \varepsilon.$$

The first inequality follows from our choice of $\varepsilon'$. The second inequality holds since $(1+\frac{x}{y})^y \leq \exp(x)$. The third inequality holds because $\exp(x^2) \leq 1 + x$ for $x \in [0, 0.7]$ (we assume $\varepsilon / \log n < 0.7$ without loss of generality).

LEMMA 6.5. *The probability that in a run of Algorithm 4 every execution of line 10 is successful is at least $1/2$.*

PROOF. Line 10 of Algorithm 4 is executed at most $Q \cdot \log n$ times, where $Q$ is an upper bound for the number of different $\varepsilon'$-signatures of solutions of instances of size at most $N$. Each execution fails with a probability of at most $\frac{1}{2Q \lfloor \log n \rfloor}$. Thus by a union bound, the probability that one of them fails is at most $1/2$. □

Since randomization is only needed for MINCC-APPROX, Lemma 6.5 completes the proof of Theorem 6.1.

*Remark* 6.6. According to Bläser et al. [2006], the algorithm of Frieze et al. [1982] can be turned into a $\frac{1}{1-\gamma}$-approximation for Min-ATSP with $\gamma$-triangle inequality for $\gamma \in [\frac{1}{2}, 1)$. An instance fulfills the $\gamma$-triangle inequality, if $w(u, v) \leq \gamma \cdot (w(u, x) + w(x, v))$ for all distinct $u, v, x$. In the same way, Algorithm 4 can be turned into a $\frac{1}{1-\gamma} + \varepsilon$ approximation for this variant of multi-criteria TSP. This improves over existing results [Manthey and Ram 2009] for $\gamma \geq 0.55$.

## 7. CONCLUDING REMARKS

We have presented approximation algorithms for almost all variants of multi-criteria TSP. The approximation ratios of our algorithms are independent of the number $k$ of criteria and come close to the currently best ratios for TSP with a single objective. Furthermore, they work for any number of criteria.

Furthermore, we have devised a deterministic 7/27 approximation for 2-Max-STSP with cubic running-time, and we proved that for 2-Max-STSP, there always exists a 1/3 approximate Pareto curve that consists of a single element.

Most approximation algorithms for multi-criteria TSP use randomness since computing approximate Pareto curves of cycle covers requires randomness. This raises the question of whether there are algorithms for multi-criteria TSP that are faster, deterministic, and achieve better approximation ratios.

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