A model reduction approach for inverse problems with operator valued data

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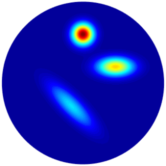
Outline

Fluorescence optical tomography: forward problem: $c \mapsto \mathtt{M}$

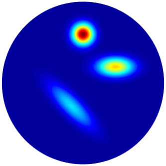


concentration

[Arridge, Schotland: Optical tomography: forward and inverse problems, Inverse Problems, 25 (2009)]

[Egger et al: On forward and inverse models in fluorescence diffuse optical tomography, IPI, 4 (2010)]

Fluorescence optical tomography: forward problem: $c \mapsto \mathtt{M}$



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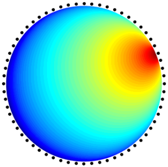

excitation

Excitation field generated by source q_j

$$
\begin{aligned}
-\nabla \cdot (\kappa_x \nabla u_{x,j}) + \mu_x u_{x,j} &= 0 \quad \text{in} \quad \Omega \\
\kappa_x \partial_n u_{x,j} + \rho_x u_{x,j} &= q_j \quad \text{on } \partial\Omega
\end{aligned}
$$

[Arridge, Schotland: Optical tomography: forward and inverse problems, Inverse Problems, 25 (2009)]

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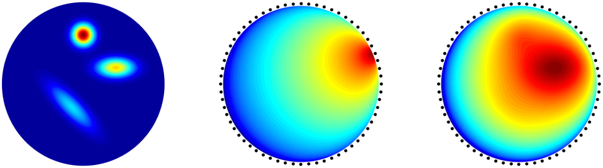

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Emission field for fluorophore concentration c and excitation field $u_{x,j}$

$$
\begin{aligned}
-\nabla\cdot(\kappa_m \nabla u_{m,j}) + \mu_m u_{m,j} &= c u_{x,j} \quad \text{in} \quad \Omega \\
\kappa_m \partial_n u_{m,j} + \rho_m u_{m,j} &= 0 \qquad \text{on } \partial\Omega
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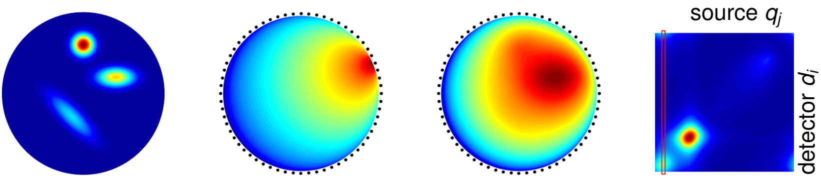

concentration excitation emission measurements

Excitation field generated by source q_j

$$\begin{aligned} -\nabla\cdot(\kappa_x \nabla u_{x,j}) + \mu_x u_{x,j} &= 0 \quad \text{in} \quad \Omega \\ \kappa_x \partial_n u_{x,j} + \rho_x u_{x,j} &= q_j \quad \text{on } \partial\Omega \end{aligned}$$

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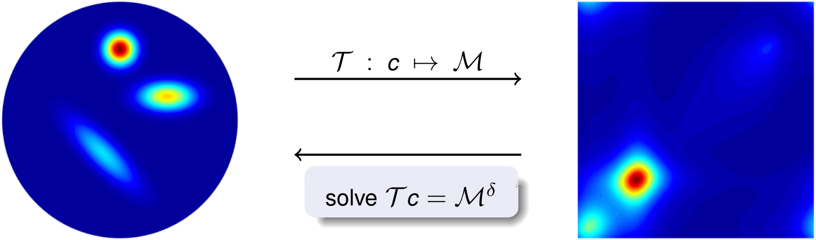
Measurements at detector (pixel) d_i: $\mathtt{M}_{ij} = \int_{\partial\Omega} d_i(x) u_{m,j}(x) d\sigma(x)$

[Arridge, Schotland: Optical tomography: forward and inverse problems, Inverse Problems, 25 (2009)]

[Egger et al: On forward and inverse models in fluorescence diffuse optical tomography, IPI, 4 (2010)]

Inverse Problem

- $\mathtt{M} \approx \mathcal{M} = \mathcal{M}(c) : q \to u_{m|\partial\Omega}$ (infinite dim.) measurement operator
- Use of $\mathcal{M}$ yields algorithms that are independent of the discretization



concentration c

measurements: $\mathcal{M}$

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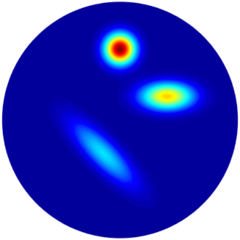
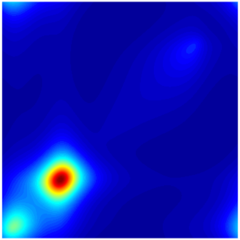


concentration c

$\mathcal{T} : c \mapsto \mathcal{M}$ (→)

(←) solve $\mathcal{T}c = \mathcal{M}^\delta$



measurements: $\mathcal{M}$

Tikhonov regularization $\|\mathcal{T}c - \mathcal{M}^\delta\|^2 + \alpha\|c\|^2 \to \min$

Regularized normal equations $(\mathcal{T}^\star\mathcal{T} + \alpha\mathcal{I})c_\alpha^\delta = \mathcal{T}^\star\mathcal{M}^\delta$

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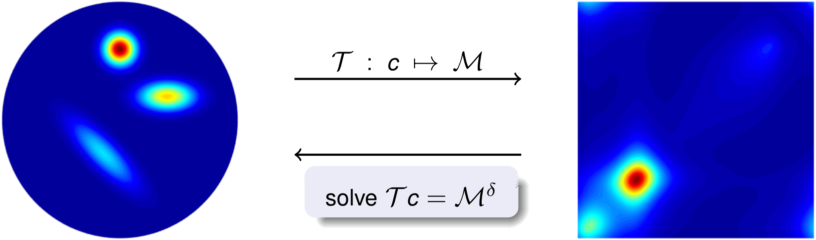

concentration c measurements: $\mathcal{M}$

Tikhonov regularization $\quad \|\mathcal{T}c - \mathcal{M}^\delta\|^2 + \alpha\|c\|^2 \to \min$

Regularized normal equations $\quad (\mathcal{T}^\star\mathcal{T} + \alpha\mathcal{I})c_\alpha^\delta = \mathcal{T}^\star\mathcal{M}^\delta$

- solve with conjugate gradients (apply $\mathcal{T}c$ and $\mathcal{T}^\star M$ (its truth approximation))
- expensive for large parameter space and many measurements

[Freiberger et al: High-performance image reconstruction in fluorescence tomography on desktop computers and graphics hardware, Biomed. Opt. Express, 2 (2011)]

Pathway to model reduction via projections

Projection

$$\mathcal{T}_N = \mathcal{Q}_N \mathcal{T}$$

Error bound

$$\|\mathcal{Q}_N \mathcal{T} - \mathcal{T}\| \leq \delta.$$

Reconstruction

$$(\mathcal{T}_N^\star \mathcal{T}_N + \alpha \mathcal{I}_N) c_{\alpha,N}^\delta = \mathcal{T}_N^\star \mathcal{M}_N^\delta$$

Reconstruction error bound

$$\|c_{\alpha,N}^\delta - c_\alpha^\delta\| \leq C(\alpha)\delta$$

[Bakushinsky, Kokurin: Iterative Methods for Approximate Solution of Inverse Problems. Springer 2004]

[Neubauer: An a posteriori parameter choice for Tikhonov regularization in the presence of modeling errors. APNUM 4 (1988)]

Offline-Online decomposition

Offline. Setup of the approximations $\mathcal{Q}_N$, $\mathcal{T}_N^\star$, and $\mathcal{T}_N \mathcal{T}_N^\star$.

Online. Computation of the regularized solution requires

step	computations	complexity	memory
compression	$\mathcal{M}_N^\delta = \mathcal{Q}_N \mathcal{M}^\delta$	Nk^2	Nk^2
analysis	$z_{\alpha,N}^\delta = g_\alpha(\mathcal{T}_N \mathcal{T}_N^\star) \mathcal{M}_N^\delta$	N^2	N^2
synthesis	$c_{\alpha,N}^\delta = \mathcal{T}_N^\star z_{\alpha,N}^\delta$	Nm	Nm

Truth approximation $\mathcal{T} \in \mathbb{R}^{m \times k^2}$

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Truth approximation $\mathcal{T} \in \mathbb{R}^{m\times k^2}$

$\mathcal{T}_N$ being a truncated SVD of $\mathcal{T}$ is the benchmark, but expensive!

see, e.g., [Hochstenbach 2001, Markel et al 2003, Stoll 2012, Chaillat et al 2012, Musco et al 2015,...]

A decomposition of the forward operator: $\mathcal{T} = \mathcal{V}'\mathcal{D}(\cdot)\mathcal{U}$

Adjoint emission problem

$$\begin{aligned} -\nabla \cdot (\kappa_m \nabla v_m) + \mu_m v_m &= 0 \quad \text{in} \quad \Omega \\ \kappa_m \partial_n v_m + \rho_m v_m &= q_m \quad \text{on } \partial\Omega \end{aligned}$$

Solution operators of excitation and adjoint emission problem

$$\mathcal{U} : q_x \mapsto \mathcal{U} q_x := u_x$$
$$\mathcal{V} : q_m \mapsto \mathcal{V} q_m := v_m.$$

Multiplication operator

$$\mathcal{D}(c)u = cu, \qquad \text{i.e.} \quad \langle \mathcal{D}(c)u, v\rangle = \int_\Omega c\, u\, v\, dx.$$

One can show that $u_{m|\partial\Omega} = \mathcal{V}'\mathcal{D}(c)u_x$ in correct functional analytic setting, i.e.,

$$\mathcal{T}c = \mathcal{V}'\mathcal{D}(c)\mathcal{U}.$$

[Egger et al: On forward and inverse models in fluorescence diffuse optical tomography, IPI, 4 (2010)]

Procedure for setting up the projection for $\mathcal{T} = \mathcal{V}'\mathcal{D}(\cdot)\mathcal{U}$

1. Accurate projections

$$\mathcal{U}_K = \mathcal{U}\mathcal{Q}_{K,\mathcal{U}} \qquad \text{and} \qquad \mathcal{V}_K = \mathcal{V}\mathcal{Q}_{K,\mathcal{V}}$$

lead to

$$\mathcal{T}_{K,K}c = \mathcal{Q}_{K,K}\mathcal{T}c = \mathcal{V}_K'\mathcal{D}(c)\mathcal{U}_K, \qquad \mathcal{M}^\delta_{K,K} = (\mathcal{Q}'_{K,\mathcal{V}}\mathcal{M}^\delta)\mathcal{Q}_{K,\mathcal{U}}$$

with error bound $\|\mathcal{T}_{K,K} - \mathcal{T}\| \leq \delta$

2. Further compression of $\mathcal{T}_{K,K}$ gives rise to desired projection

$$\mathcal{Q}_N = \mathcal{P}_N\mathcal{Q}_{K,K}, \qquad \mathcal{M}^\delta_N = \mathcal{P}_N\mathcal{M}^\delta_{K,K}$$

Remark. Computation of $\mathcal{M}^\delta_{K,K} = (\mathcal{Q}'_{K,\mathcal{V}}\mathcal{M}^\delta)\mathcal{Q}_{K,\mathcal{U}}$ requires only $O(Kk + K^2)$ mem instead of $O(Nk^2)$ if tensor structure is used.

Related to step 1.: [Herrmann et al 2009, Krebs et al 2009, Roosta-Khorasani et al 2014, Markel et al 2019]

Fluorescence diffuse optical tomography

- Model
- Inverse problem
- Outline of model reduction approach

Abstract Analysis: Model reduction for $\mathcal{T}c = \mathcal{V}'\mathcal{D}(c)\mathcal{U}$

- Properties of the forward operator
- Step 1: Tensor product approximation
- Step 2: Quasi-optimal compression

Model reduction in action: Application to FDOT

- Truth approximation and implementation
- Complexity estimates
- Runtimes and ranks

Conclusion

Abstract setting

Assumptions. $\mathbb{U}, \mathbb{V}, \mathbb{X}, \mathbb{Y}, \mathbb{Z}$ separable Hilbert spaces

$$\mathcal{U} \in \mathbb{HS}(\mathbb{Y}, \mathbb{U}), \qquad \mathcal{V} \in \mathbb{HS}(\mathbb{Z}, \mathbb{V}), \quad \mathcal{D} \in \mathcal{L}(\mathbb{X}, \mathcal{L}(\mathbb{U}, \mathbb{V}'))$$

---

recall: Any $\mathcal{S} \in \mathcal{L}(\mathbb{A}, \mathbb{B})$ compact has singular value decomposition (SVD)

$$\mathcal{S}a = \sum_{k=1}^{\infty} (a, a_k)_{\mathbb{A}} \sigma_{k,\mathcal{S}} b_k$$

with ONBs $\{a_k\} \subset \mathbb{A}$ and $\{b_k\} \subset \mathbb{B}$. Truncated SVD

$$\mathcal{S}_K a = \sum_{k=1}^{K} (a, a_k)_{\mathbb{A}} \sigma_{k,\mathcal{S}} b_k.$$

Error $\|\mathcal{S} - \mathcal{S}_K\|_{\mathcal{L}(\mathbb{A},\mathbb{B})} = \sigma_{K+1,\mathcal{S}}$,

$\mathcal{S} \in \mathbb{HS}(\mathbb{A}, \mathbb{B})$ iff $\{\sigma_{k,\mathcal{S}}\}_k \in \ell_2$, and $\|\mathcal{S}\|^2_{\mathbb{HS}(\mathbb{A},\mathbb{B})} = \sum_{k=1}^{\infty} \sigma^2_{k,\mathcal{S}}$

---

see, e.g., [Hackbusch: Tensor spaces and numerical tensor calculus. Springer. 2014]

Properties of the forward operator

$\mathbb{U}, \mathbb{V}, \mathbb{X}, \mathbb{Y}, \mathbb{Z}$ separable Hilbert spaces

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Lemma. $\mathcal{T}(c) = \mathcal{V}'\, \mathcal{D}(c)\mathcal{U}$ defines a bounded linear compact operator $\mathcal{T} : \mathbb{X} \to \mathbb{HS}(\mathbb{Y}, \mathbb{Z}')$.

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Proof (sketch).

- Consider TSVDs: $\mathcal{U}_K = \mathcal{U} \mathcal{Q}_{K,\mathcal{U}}$ and $\mathcal{V}_K = \mathcal{V} \mathcal{Q}_{K,\mathcal{V}}$ with rank K such that

$$\|\mathcal{U} - \mathcal{U}_K\|_{\mathcal{L}(\mathbb{Y},\mathbb{U})} \lesssim K^{-1/2} \qquad \text{and} \qquad \|\mathcal{V} - \mathcal{V}_K\|_{\mathcal{L}(\mathbb{Z},\mathbb{V})} \lesssim K^{-1/2}$$

- Define $\mathcal{T}_{K,K} : \mathbb{X} \to \mathbb{HS}(\mathbb{Y}, \mathbb{Z}')$ by $\mathcal{T}_{K,K} c = \mathcal{V}_K' \mathcal{D}(c) \mathcal{U}_K$
- rank $\mathcal{T}_{K,K} \leq K^2$.
- Show $\|\mathcal{T} - \mathcal{T}_{K,K}\|_{\mathcal{L}(\mathbb{X},\mathbb{HS}(\mathbb{Y},\mathbb{Z}'))} \lesssim K^{-1/2}$ via triangle inequality.

Step 1: Tensor product approximation of $\mathcal{T}$

Corollary. For any $\delta > 0$ there exists $K \in \mathbb{N}$ with $K \lesssim \delta^{-2}$ such that

$$\|\mathcal{U} - \mathcal{U}_K\|_{\mathcal{L}(\mathbb{Y},\mathbb{U})} \le \delta \quad \text{and} \quad \|\mathcal{V} - \mathcal{V}_K\|_{\mathcal{L}(\mathbb{Z},\mathbb{V})} \le \delta,$$

and $\mathcal{T}_{K,K}c = \mathcal{V}_K' \mathcal{D}(c) \mathcal{U}_K$ satisfies

$$\|\mathcal{T} - \mathcal{T}_{K,K}\|_{\mathcal{L}(\mathbb{X},\mathbb{HS}(\mathbb{Y},\mathbb{Z}'))} \lesssim \delta.$$

If $\sigma_{k,\mathcal{U}}, \sigma_{k,\mathcal{V}} \lesssim k^{-\alpha}$ for $\alpha > 1/2$, then $K \simeq \delta^{-1/\alpha}$ and $\mathrm{rank}(T_{K,K}) \lesssim \delta^{-2/\alpha}$.

Compare to: [Markel et al: Fast linear inversion for highly overdetermined inverse scattering problems, Inverse Problems, 35 (2019)]

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and $\mathcal{T}_{K,K} c = \mathcal{V}_K' \mathcal{D}(c) \mathcal{U}_K$ satisfies

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Can we do better?

Compare to: [Markel et al: Fast linear inversion for highly overdetermined inverse scattering problems, Inverse Problems, 35 (2019)]

Step 1': Hyperbolic cross approximation of $\mathcal{T}$

Lemma. Let $\sigma_{k,\mathcal{U}} \lesssim k^{-\beta}$ and $\sigma_{k,\mathcal{V}} \lesssim k^{-\alpha}$ for $\beta > 1/2$ and $\alpha > \beta + 1/2$. Then for any $\delta > 0$, we can construct $\mathcal{T}_{\widehat{K}}$ with rank $\lesssim \delta^{-1/\beta}$:

$$\|\mathcal{T} - \mathcal{T}_{\widehat{K}}\|_{\mathcal{L}(\mathbb{X},\mathbb{HS}(\mathbb{Y},\mathbb{Z}'))} \lesssim \delta.$$

cf. [Dung et al: Hyperbolic cross approximation, Birkhäuser/Springer, Cham, 2018.]

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Remarks.
(i) rank $\mathcal{T}_{K,K} = K^2 \simeq \delta^{-2/\alpha}$, with $\mathcal{T}_{K,K}c = \mathcal{V}_K' \mathcal{D}(c)\mathcal{U}_K$
(ii) rank $\mathcal{T}_{\widehat{K}} \lesssim \delta^{-1/(\alpha-(1/2+\epsilon))} \implies$ rank $\mathcal{T}_{K,K}$ is not optimal if $\alpha > 1$
(iii) $\mathcal{T}_{\widehat{K}}$ can be realized as a hyperbolic cross approximation of $\mathcal{T}_{K,K}$.

cf. [Dung et al: Hyperbolic cross approximation, Birkhäuser/Springer, Cham, 2018.]

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(i) rank $\mathcal{T}_{K,K} = K^2 \simeq \delta^{-2/\alpha}$, with $\mathcal{T}_{K,K}c = \mathcal{V}'_K \mathcal{D}(c)\mathcal{U}_K$
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(iii) $\mathcal{T}_{\widehat{K}}$ can be realized as a hyperbolic cross approximation of $\mathcal{T}_{K,K}$.

Proof (sketch). Let $\{\sigma_{k,*}, a_{k,*}, b_{k,*}\}$ denote the singular systems for $\mathcal{U}$ and $\mathcal{V}'$, respectively. The hyperbolic cross approximation

$$\mathcal{T}_{\widehat{K}}(c) = \sum_{k\geq 1} \sum_{\ell=1}^{L_k} \sigma_{\ell,\mathcal{U}}\, \sigma_{k,\mathcal{V}'}\, (\cdot, a_{\ell,\mathcal{U}})_{\mathbb{Y}}\, \langle \mathcal{D}(c) b_{\ell,\mathcal{U}}, a_{k,\mathcal{V}'}\rangle_{\mathbb{V}'\times\mathbb{V}}\, b_{k,\mathcal{V}'},$$

with the choice $L_k = \lfloor \widehat{K}/k^{1+\epsilon} \rfloor$, $\widehat{K} \simeq \delta^{-1/\beta}$, and $\epsilon = (\alpha - \beta - 1/2)/(2\beta) > 0$ has the required properties.

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cf. [Dung et al: Hyperbolic cross approximation, Birkhäuser/Springer, Cham, 2018.]

Step 2: Quasi-optimal low-rank approximation via TSVD

Lemma. Let $\delta > 0$ and let $\mathcal{T}^\delta : \mathbb{X} \to \mathbb{HS}(\mathbb{Y}, \mathbb{Z}')$ be a linear compact operator such that $\|\mathcal{T}^\delta - \mathcal{T}\|_{\mathcal{L}(\mathbb{X},\mathbb{HS}(\mathbb{Y},\mathbb{Z}'))} \le C\delta$ for some $C > 0$. Let $\mathcal{P}^\delta_{N^\delta}\mathcal{T}^\delta$ denote the truncated singular value decomposition of $\mathcal{T}^\delta$ with minimal rank N^δ such that

$$\|\mathcal{T}^\delta - \mathcal{P}^\delta_{N^\delta}\mathcal{T}^\delta\|_{\mathcal{L}(\mathbb{X},\mathbb{HS}(\mathbb{Y},\mathbb{Z}'))} \le (C+1)\delta.$$

Then $N^\delta \le N^{\mathrm{svd}}$, the rank of the TSVD of $\mathcal{T}$ that yields a δ error, and

$$\|\mathcal{T} - \mathcal{P}^\delta_{N^\delta}\mathcal{T}^\delta\|_{\mathcal{L}(\mathbb{X},\mathbb{HS}(\mathbb{Y},\mathbb{Z}'))} \le (2C+1)\delta,$$

i.e., $\mathcal{P}^\delta_{N^\delta}\mathcal{T}^\delta$ is a δ-approximation for $\mathcal{T}$ with quasi-optimal rank.

Proof: Step 1. Perturbation of singular values

Claim. (cf [Kato1966]): Let $\|\mathcal{T}^\delta - \mathcal{T}\| \leq C\delta$. For each $k \in \mathbb{N}$ one has

$$\sigma_k - C\delta \leq \sigma_k^\delta \leq \sigma_k + C\delta,$$

where $\{\sigma_k\}$ and $\{\sigma_k^\delta\}$ denote the singular values of $\mathcal{T}$ and $\mathcal{T}^\delta$, respectively.

Choose $\varepsilon > 0$, and let $\mathcal{P}_M^{\text{svd}}\mathcal{T}$ denote the TSVD of $\mathcal{T}$ with optimal rank M s.t.

$$\|\mathcal{P}_M^{\text{svd}}\mathcal{T} - \mathcal{T}\| \leq \varepsilon.$$

Optimality of M and the non-expansiveness of the projection implies

$$\begin{aligned}\sigma_{M+1}^\delta = \|(\mathcal{I} - \mathcal{P}_M^\delta)\mathcal{T}^\delta\| &\leq \|(\mathcal{I} - \mathcal{P}_M^{\text{svd}})\mathcal{T}^\delta\| \\ &\leq \|(\mathcal{I} - \mathcal{P}_M^{\text{svd}})\mathcal{T}\| + \|(\mathcal{I} - \mathcal{P}_M^{\text{svd}})(\mathcal{T} - \mathcal{T}^\delta)\| \leq \varepsilon + C\delta.\end{aligned}$$

For $\varepsilon = \sigma_{k+1}$, we have $M = M(\varepsilon) = k$, and we conclude that

$$\sigma_{k+1}^\delta \leq \sigma_{k+1} + C\delta.$$

The second inequality follows by interchanging the roles of $\mathcal{T}$ and $\mathcal{T}^\delta$.

Proof: Step 2.

Let N^δ be as in the lemma: $\sigma^\delta_{N^\delta+1} \leq (C+1)\delta < \sigma^\delta_{N^\delta}$.

Let $N^{\text{svd}} = M(\delta)$ as defined in Step 1: $\sigma_{N^{\text{svd}}+1} \leq \delta$.

The claim implies

$$\sigma^\delta_{N^{\text{svd}}+1} \leq \sigma_{N^{\text{svd}}+1} + C\delta \leq (C+1)\delta$$

Monotonicity of the singular values: $N^\delta \leq N^{\text{svd}}$.

Finally,

$$\|\mathcal{P}^\delta_{N^\delta}\mathcal{T}^\delta - \mathcal{T}\| \leq \|\mathcal{P}^\delta_{N^\delta}\mathcal{T}^\delta - \mathcal{T}^\delta\| + \|\mathcal{T}^\delta - \mathcal{T}\| \leq (2C+1)\delta,$$

i.e., $\mathcal{P}^\delta_{N^\delta}\mathcal{T}^\delta$ is a δ-approximation for $\mathcal{T}$ with quasi-optimal rank $N^\delta \leq N^{\text{svd}}$.

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Function space setting

recall: Assumptions of abstract theory: $\mathbb{U}, \mathbb{V}, \mathbb{X}, \mathbb{Y}, \mathbb{Z}$ separable Hilbert spaces,

$$\mathcal{U} \in \mathbb{HS}(\mathbb{Y}, \mathbb{U}), \qquad \mathcal{V} \in \mathbb{HS}(\mathbb{Z}, \mathbb{V}), \quad \mathcal{D} \in \mathcal{L}(\mathbb{X}, \mathcal{L}(\mathbb{U}, \mathbb{V}'))$$

Solution operators

$$\begin{aligned} \mathcal{U} : H^1(\partial\Omega) \to H^1(\Omega), &\quad q_x \mapsto \mathcal{U} q_x := u_x \\ \mathcal{V} : H^1(\partial\Omega) \to H^1(\Omega), &\quad q_m \mapsto \mathcal{V} q_m := v_m. \end{aligned}$$

Multiplication operator

$$\mathcal{D} : L^2(\Omega) \to \mathcal{L}(H^1(\Omega), H^1(\Omega)'), \quad \mathcal{D}(c)u = cu.$$

Function spaces

$$\mathbb{U} = \mathbb{V} = H^1(\Omega), \mathbb{Y} = \mathbb{Z} = H^1(\partial\Omega), \mathbb{X} = L^2(\Omega).$$

Lemma. The operators $\mathcal{U}$ and $\mathcal{V}$ are Hilbert-Schmidt and their singular values decay like $\sigma_{k,\mathcal{U}} \lesssim k^{-3/(2d-2)}$ and $\sigma_{k,\mathcal{V}} \lesssim k^{-3/(2d-2)}$.

Truth approximation via standard FEM

- T_h a quasi-uniform conforming triangulation of the domain Ω
- $\mathbb{P}_1$-Lagrange finite elements: $\mathbb{U}_h, \mathbb{V}_h \subset H^1(\Omega)$, $\mathbb{X}_h \subset \mathbb{X}$; $\dim = m \approx h^{-d}$
- induced boundary finite elements: $\mathbb{Y}_h, \mathbb{Z}_h \subset H^1(\partial\Omega)$; $\dim = k \approx h^{-d+1}$

$\mathtt{U} \in \mathbb{R}^{m\times k}$ discrete counterpart of the operator $\mathcal{U}$:

$$(\mathtt{K_x} + \mathtt{M_x} + \mathtt{R_x})\,\mathtt{U} = \mathtt{E_x}\mathtt{Q_x}.$$

$\mathtt{V} \in \mathbb{R}^{m\times k}$ discrete counterpart of the operator $\mathcal{V}$:

$$(\mathtt{K_m} + \mathtt{M_m} + \mathtt{R_m})\,\mathtt{V} = \mathtt{E_m}\mathtt{Q_m}.$$

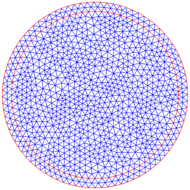
Algebraic form of the truth approximation

$$\mathtt{T}(\mathtt{c}) = \mathtt{V}^\top\,\mathtt{D}(\mathtt{c})\,\mathtt{U},$$

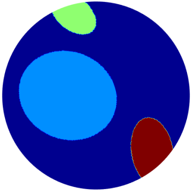
Discrete measurement

$$\mathtt{M}_{ij} = (\mathtt{V}^\top \mathtt{D}(\mathtt{c})\mathtt{U})_{ij} = \mathtt{V}(:,i)^\top \mathtt{D}(\mathtt{c})\mathtt{U}(:,j).$$

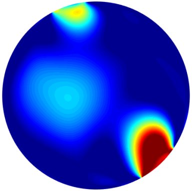
Numerical example: setup



Computational domain and coarse mesh



Minimum norm solution $c^\dagger$



Reconstruction c_α^δ for $\delta = 10^{-5}$

Dimensions: $m = \dim(\mathbb{X}_h) = \dim(\mathbb{U}_h) = \dim(\mathbb{V}_h)$ and $k = \dim(\mathbb{Y}_h) = \dim(\mathbb{Z}_h)$
discretization error: $de_h = \|\mathcal{T}_h - \mathcal{T}\|_{\mathcal{L}(\mathbb{X},\mathbb{HS}(\mathbb{Y},\mathbb{Z}'))}$

ref	0	1	2	3	4	5
m	993	3 881	15 345	61 025	243 393	927 161
k	88	176	352	704	1 408	2 816
de_h	$6.31 \cdot 10^{-4}$	$1.69 \cdot 10^{-4}$	$4.34 \cdot 10^{-5}$	$1.10 \cdot 10^{-5}$	$2.75 \cdot 10^{-6}$	—

Remark. For *ref* $= 5$, $\mathtt{T} : \mathbb{R}^{927\,161} \to \mathbb{R}^{2\,816\times 2\,816}$; storage 56TB of memory; one single evaluation of $\mathtt{T}(\mathtt{c})$ require approximately 7Tflops.

Memory and operation cost

Offline

- Compression of $\mathtt{U}, \mathtt{V} \in \mathbb{R}^{m \times k}$: mem $O(km)$, ops $O(k^2 m + k^3)$
  These steps are at most as expensive as one application of $\mathtt{T}$.
- Compression: mem $O(\mathtt{K}m)$
  - tensor product $\mathtt{TKK}$: ops $O(\mathtt{K}^2 m)$
  - hyperbolic cross $\mathtt{TK}$: ops $O(m\mathtt{K} \ln \mathtt{K})$
- Recompression to obtain $\mathtt{TN}$ with quasi-optimal rank $\mathtt{N}$
  - of tensor product: ops $O(m\mathtt{K}^4 + \mathtt{K}^6)$
  - of hyperbolic cross: ops $O(m(\mathtt{K} \ln \mathtt{K})^2 + (\mathtt{K} \ln \mathtt{K})^4)$

  Use hyperbolic cross approximation $\mathtt{TK}$ to compute $\mathtt{TN}$.

Online (inverse problem)

- Data compression: mem $O(\mathtt{K}k)$, ops $O(\mathtt{K}^2 k + \mathtt{K}k^2)$
- Analysis: depends on $\mathtt{N}$ only (cheap!)
- Synthesis: mem $O(m\mathtt{N})$, ops $O(m\mathtt{N})$

Truncation ranks N and timings for SVD

of T (full operator), TKK (tensor product approx.) and TK (hyperbolic cross approx.)

refinements	0	1	2	3	4	5
svd(T) in sec	6.46	28.23	284.33	—	—	—
N(T)	231	303	473	—	—	—
svd(TKK) in sec	6.45	15.05	48.40	248.42	994.66	—
N(TKK)	231	276	296	310	314	—
setup of TK, TKTKt in sec	0.01	0.013	1.38	6.31	30.48	140.32
rank(TK)	403	933	1 725	1 867	1 905	1 917
svd(TK) in sec	0.08	0.87	2.57	3.51	3.87	4.03
N(TK)	166	266	391	396	401	403

Timings and error $\|c_\alpha^\delta - c^\dagger\|$ for solving the inverse problem

Full operator ($\mathtt{T}$) and tensor product approximation ($\mathtt{TKK}$)

refinements	0	1	2	3	4	5
time($\mathtt{T}$) in sec	1.24	13.91	320.73	—	—	—
time($\mathtt{TKK}$) in sec	1.22	10.07	65.02	382.76	—	—

Reduced order model

ref	0	1	2	3	4	5
data compression	0.001	0.005	0.028	0.114	0.457	1.831
regularized normal equations	0.002	0.001	0.002	0.003	0.003	0.003
synthesis	0.001	0.001	0.004	0.015	0.061	0.107
reconstruction error	0.112	0.108	0.107	0.107	0.107	0.107

Fluorescence diffuse optical tomography
- Model
- Inverse problem
- Outline of model reduction approach

Abstract Analysis: Model reduction for $\mathcal{T}c = \mathcal{V}'\mathcal{D}(c)\mathcal{U}$
- Properties of the forward operator
- Step 1: Tensor product approximation
- Step 2: Quasi-optimal compression

Model reduction in action: Application to FDOT
- Truth approximation and implementation
- Complexity estimates
- Runtimes and ranks

Conclusion

Conclusion and final remarks

- Derived a systematic way to obtain a certified reduced order model of quasi-optimal rank for linear operators of the form $\mathcal{T}c = \mathcal{V}'\mathcal{D}(c)\mathcal{U}$
- Advantages:
  - Fast setup time (cost $=$ one single evaluation of forward operator)
  - Partial compression during recording (access to full data is never required)

- Problems with a similar structure $\mathcal{T} = \mathcal{V}'\mathcal{D}(\cdot)\mathcal{U}$:
  - Inverse scattering [Colton & Kress, Grinberg & Kirsch, Somersalo et al 1992]
  - Aeroacoustic source problems [Hohage et al 2020]
- Compression of $\mathcal{U}$ and $\mathcal{V}$ is related to
  - optimal sources and detectors [Herrmann et al 2009, Krebs et al 2009, van den Doel 2012, Roosta-Khorasani et al 2014]
  - optimal experimental design [Pukelsheim 2006]

[J. Dölz, H. Egger, M. Schlottbom: A model reduction approach for inverse problems with operator valued data `https://arxiv.org/abs/2004.11827)`]